



Research article

Derivation of Bessel function closed-form solutions in zero dimensional φ^4 -field theoryRanjiva M. Munasinghe^{a,b}^a MIND Analytics & Management, 10/1 De Fonseka Place, Colombo 5, Sri Lanka^b Sri Lanka Institute of Information Technology, SLIIT Malabe Campus, New Kandy Road, Malabe, 10115, Sri Lanka

ARTICLE INFO

Keywords:

0-dimensional φ^4 -field theory
 0-dimensional quartic anharmonic oscillator
 Solvable quantum field theory
 Closed-form solution
 Bessel functions
 Perturbation theory

ABSTRACT

The integral $\int_{-\infty}^{\infty} e^{-x^2-gx^4} dx$ is used as an introductory learning tool in the study of Quantum Field Theory and path integrals. Typically, it is analyzed via perturbation theory. Closed-form solutions have been quoted for which I could not find any derivation. Using a simple and elegant transformation, the close form solutions for the integral and its even positive integer moments can be obtained in terms of Bessel functions.

1. Introduction

A common integral in the preparatory studies of Quantum Field Theory (QFT), perturbation theory¹ and Feynman diagrams is the "toy model":

$$Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2-gx^4} dx \quad g \geq 0. \quad (1)$$

The function $Z(g)$ is referred to as the partition function of i) *zero dimensional φ^4 -field theory* 1,2 or alternatively ii) *the zero-dimensional anharmonic oscillator* [3]. The references [2,3] state (without derivation) the closed-form solution to the integral given by (1):

$$Z(g) = \frac{1}{\sqrt{4\pi g}} \exp\left[\frac{1}{8g}\right] K_{1/4}\left(\frac{1}{8g}\right) \quad (2)$$

Note the *modified Bessel function of the second kind* K_ν , also called a MacDonald function [2], in equation (2). The function can be expressed as [4] for $|\arg z| < \pi/2$, i.e. $\text{Re } z > 0$:

$$K_\nu(z) = \int_0^\infty \cosh(\nu t) e^{-z \cosh t} dt. \quad (3)$$

The expression (2) is in agreement with the formula (Ch. 3.323 No. 3) in Ref. [5], which in turn refers to the formula (Ch. 4.5 No. 34) in Ref. [6]. No derivations of the formulae are stated in either [5] or [6]. In addition, we note the alternative formulation of the closed-form solution for $Z(g)$ in Ref. [2]:

E-mail address: ranjiva@minklanka.com.

¹ Please refer to AppendixB.

$$Z(g) = \left(\frac{1}{2g}\right)^{1/4} \exp\left[\frac{1}{8g}\right] D_{-1/2}\left(\frac{1}{2g}\right). \tag{4}$$

Equation (4) can be verified from (2) using the identity [7]:

$$D_{-1/2}(z) = \sqrt{\frac{z}{2\pi}} K_{1/4}\left(\frac{z^2}{4}\right)$$

Derivation.

We re-cast (1) as

$$Z(g) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2 - gx^4} dx, \tag{5}$$

and define a new variable transformation

$$x = \frac{1}{\sqrt{g}} \sinh\left(\frac{\xi}{4}\right) \tag{6}$$

$$dx = \frac{1}{4\sqrt{g}} \cosh\left(\frac{\xi}{4}\right) d\xi \tag{7}$$

When switching to the variable ξ , the limits in the integral in (5) are unchanged. Next, we consider the exponent in (5):

$$\begin{aligned} x^2 + gx^4 &= x^2 \bullet [1 + gx^2] = g^{-1} \sinh^2\left(\frac{\xi}{4}\right) \bullet \left[1 + \sinh^2\left(\frac{\xi}{4}\right)\right] = g^{-1} \sinh^2\left(\frac{\xi}{4}\right) \bullet \cosh^2\left(\frac{\xi}{4}\right) = (4g)^{-1} \\ &\bullet \sinh^2\left(\frac{\xi}{2}\right) = (4g)^{-1} \left[\cosh^2\left(\frac{\xi}{2}\right) - 1\right] = (4g)^{-1} \left[\frac{1}{2}(\cosh \xi + 1) - 1\right] = \frac{1}{8g} \cosh \xi - \frac{1}{8g}. \end{aligned} \tag{8}$$

Using (6), (7), and (8), we re-write (5) as

$$Z(g) = \frac{1}{\sqrt{4\pi g}} \exp\left[\frac{1}{8g}\right] \int_0^\infty \cosh\left(\frac{\xi}{4}\right) e^{-\frac{1}{8g} \cosh \xi} d\xi. \tag{9}$$

The final step is to use the definition (3) to substitute for the integral in (9) to obtain the expression (2):

$$Z(g) = \frac{1}{\sqrt{4\pi g}} \exp\left[\frac{1}{8g}\right] K_{1/4}\left(\frac{1}{8g}\right).$$

This is in agreement with the formula (Ch. 3.323 No. 3) in Ref. [5].

1.1 Closed form expression for the moments

The integrand in the partition function (1) is an even function and should thus only have even moments given by the formula:

$$\langle x^{2n} \rangle = \frac{1}{Z(g)} \frac{2}{\sqrt{\pi}} \int_0^\infty x^{2n} e^{-x^2 - gx^4} dx. \tag{10}$$

We proceed as before using the variable transformations (6), (7) and (3) to re-write (10):

$$\langle x^{2n} \rangle = \mathcal{N} \int_0^\infty \sinh^{2n}\left(\frac{\xi}{4}\right) \cosh\left(\frac{\xi}{4}\right) e^{-\frac{1}{8g} \cosh \xi} d\xi. \tag{11}$$

The normalization factor $\mathcal{N} = \mathcal{N}(n, g)$ can be expressed as:

$$\mathcal{N}(n, g) = \frac{1}{Z(g)} \frac{e^{1/8g}}{\sqrt{4\pi g^{2n+1}}} = \left[g^n \bullet K_{1/4}\left(\frac{1}{8g}\right)\right]^{-1}$$

To proceed we make use of

$$\sinh^{2n} x = [\cosh^2 x - 1]^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \cosh^{2k} x,$$

to transform (11) to

$$\langle x^{2n} \rangle = \mathcal{N} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \int_0^\infty \cosh^{2k+1}\left(\frac{\xi}{4}\right) e^{-\frac{1}{8g} \cosh \xi} d\xi. \tag{12}$$

We first use the trigonometric identity for odd powers of cosine [8] and then apply Osborn’s rule [9] to convert the identity to the hyperbolic analogue:

$$\cosh^{2k+1} x = \frac{1}{4^k} \sum_{m=0}^k \binom{2k+1}{m} \cosh([2k+1-2m] \bullet x).$$

We can now finish up (12):

$$\langle x^{2n} \rangle = \mathcal{N}' \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{n-k}}{4^k} \sum_{m=0}^k \binom{2k+1}{m} \int_0^\infty \cosh\left(\frac{[2(k-m)+1] \bullet \xi}{4}\right) e^{-\frac{1}{8g} \cosh \xi} d\xi,$$

and simplify the integral above using (3)

$$\langle x^{2n} \rangle = \mathcal{N}' \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{n-k}}{4^k} \sum_{m=0}^k \binom{2k+1}{m} K_{\frac{2(k-m)+1}{4}}\left(\frac{1}{8g}\right). \tag{13}$$

In general, we see that (13) is of the form

$$\langle x^{2n} \rangle = \left[(4g)^n \bullet K_{1/4}\left(\frac{1}{8g}\right) \right]^{-1} \sum_{k=0}^n c_m^{(n)} K_{\frac{2m+1}{4}}\left(\frac{1}{8g}\right), \tag{14}$$

where with a little work we see that the coefficients $c_m^{(n)}$ in (14) can be expressed as:

$$c_m^{(n)} = \sum_{k=m}^n (-1)^{n-k} 4^{n-k} \binom{n}{k} \binom{2k+1}{k-m} \tag{15}$$

The coefficients in (15) can be simplified when starting from the top in descending order - for example the first few entries are

$$c_n^{(n)} = 1, c_{n-1}^{(n)} = 1 - 2n, c_{n-2}^{(n)} = n(2n-3), c_{n-3}^{(n)} = \frac{n(1-2n)(2n-5)}{3}$$

The even moments $2n$ for $n = 1, 2, 3$ are then given by:

$$\langle x^2 \rangle = \frac{1}{4g} \left[\frac{K_{3/4}\left(\frac{1}{8g}\right)}{K_{1/4}\left(\frac{1}{8g}\right)} - 1 \right] \quad \langle x^4 \rangle = \frac{1}{16g^2} \left[\frac{K_{5/4}\left(\frac{1}{8g}\right) - 3K_{3/4}\left(\frac{1}{8g}\right)}{K_{1/4}\left(\frac{1}{8g}\right)} + 2 \right] \quad \langle x^6 \rangle = \frac{1}{64g^3} \left[\frac{K_{7/4}\left(\frac{1}{8g}\right) - 5K_{5/4}\left(\frac{1}{8g}\right) + 9K_{3/4}\left(\frac{1}{8g}\right)}{K_{1/4}\left(\frac{1}{8g}\right)} - 5 \right]$$

2. Discussion

As I was unable to find a derivation of the closed form expressions for ϕ^4 -field theory in zero dimensions, I set about deriving the expression on my own. Along the way the trick I used to derive the expression also enables one to write a closed form expression for the even positive integer moments. I hope these results can lead to further insights on resummation methods used in the perturbative approach explored in the works [1–3,10] and references therein. I also discovered an erratum in one of the quoted formulas [6] for which the correction is mentioned in appendix. I hope this short letter will be a useful reference for practitioners and students of field theory and statistical physics.

Author contribution statement

Ranjiva M. Munasinghe: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

No data was used for the research described in the article.

Declaration of interest’s statement

The authors declare no competing interests.

Appendices.

A Erratum in Erdélyi et al.

The formula in Ref. [6] (Ch. 4.5 No. 34) states that

$$\int_0^\infty (2t)^{-3/4} e^{-2a^{1/2}t^{1/2}} e^{-pt} dt = \left[\frac{a}{2p} \right]^{1/2} \exp \left[\frac{a}{2p} \right] K_{1/4} \left(\frac{a}{2p} \right) \quad (16)$$

We also note the conditions for (16) are stated as $|\arg a| < \pi$ and $\operatorname{Re} p > 0$ [6].

We start by using the substitution $t = x^4$ to transform the LHS of (16) to:

$$2^{5/4} \int_0^\infty e^{-2a^{1/2}x^2 - px^4} dx$$

We now use a modified version of the transformation in (6)

$$x = \frac{4a^{1/2}}{p} \sinh \left(\frac{\xi}{4} \right),$$

which leads to the correct version of (16):

$$\int_0^\infty (2t)^{-3/4} e^{-2a^{1/2}t^{1/2}} e^{-pt} dt = \left[\frac{a}{2p^2} \right]^{1/4} \exp \left[\frac{a}{2p} \right] K_{1/4} \left(\frac{a}{2p} \right)$$

B Perturbative Treatment

Perturbative expansions for $Z(g)$ in (1) can be derived by expanding the exponential in the integral (1) and interchanging the order of the resulting summation and integration. In the *weak coupling* limit $g \rightarrow 0$ one obtains the divergent asymptotic expansion [1–3]:

$$Z(g) \sim \sum_{n=0}^N (-1)^n \frac{\Gamma(2n+1/2)}{n! \sqrt{\pi}} g^n \quad (17)$$

In the *strong coupling* limit $g \rightarrow \infty$ we obtain the convergent expansion [3]:

$$Z(g) \sim g^{-1/4} \sum_{n=0}^N (-1)^n \frac{\Gamma(n/2+1/4)}{2n! \sqrt{\pi}} g^{-n/2} \quad (18)$$

Both expansions, (17) and (18), can also be obtained from (1) using the appropriate expansion of $K_\nu(z)$ [2].

References

- [1] J.C. Collins, *Renormalization: an Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*, Cambridge University Press, Cambridge, 1984.
- [2] V. Yukalov, *Interplay between Approximation Theory and Renormalization Group* 50 (2) (2019).
- [3] H. Kleinert, *Converting Divergent Weak-Coupling into Exponentially Fast Convergent Strong-Coupling Expansions*, 2010.
- [4] G. Watson, *A Treatise on the Theory of Bessel Functions*, second ed., Cambridge University Press, Cambridge, 1995, p. 181.
- [5] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, seventh ed., Elsevier, Amsterdam, 2007, p. 337.
- [6] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Tables of Integral Transforms*, vol. 1, Mc-Graw Hill, New York, 1954, p. 147.
- [7] E.W. Weisstein, *Parabolic cylinder function* [Online]. Available: <https://mathworld.wolfram.com/ParabolicCylinderFunction.html>.
- [8] E.W. Weisstein, *Trigonometric power formulas*, " [Online]. Available: <https://mathworld.wolfram.com/TrigonometricPowerFormulas.html>.
- [9] E.W. Weisstein, *Osborn's rule* [Online]. Available: <https://mathworld.wolfram.com/OsbornsRule.html>.
- [10] V. Meden, *Lecture Notes: Functional Renormalization Group*, 2003 [Online]. Available: <https://www.statphys.rwth-aachen.de/cms/Statphys/Forschung/AG-Meden/~qygs/Vorlesung-Functional-Renormalization-G/lidx/1/>.