Received: 20 December 2023 *Accepted: 10 May 2024*

An Alternative Proof of Ptolemy's Theorem and its Variations

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Abstract

This paper introduces a pure geometric proof for Ptolemy's Theorem, without using trigonometry, coordinate geometry, complex numbers, vectors or any other geometric inversion techniques focusing on cyclic quadrilaterals and employing a generalized identity in relation to a cevian of an arbitrary Euclidean plane triangle. Additionally, the paper provides proofs to the converse of Ptolemy's Theorem to which almost no pure geometric complete proof is available, and to the standard Ptolemy's Inequality, to fulfil the research gap in the proofs to some extent. It also includes applications, new corollaries, derived from Ptolemy's Theorem and its converse.

Keywords: Cyclic quadrilaterals, Equilateral triangles, Inequalities, Mathematical logic, Perpendiculars, Similar triangles.

Introduction

The Ptolemy's Theorem of Cyclic Quadrilaterals founded and proved by Claudius Ptolemaeus who was an eminent Greek Mathematician, has been one of the prominent and exciting results in a geometry of a circle, throughout way back centuries ago, even at present not only in Advanced Geometry, but also in the other related sciences. There have been several alternative proofs for the Ptolemy's Theorem of cyclic quadrilaterals in the mathematics literature, using some geometric, trigonometric and non-geometric (Complex number algebra, Vector Algebra) approaches. Amarasinghe (2013) published a concise elementary proof for the Ptolemy's Theorem using purely Euclidean Geometry (without using trigonometry, vector algebra, complex numbers or any other inversion techniques), proving some other useful properties in a cyclic quadrilateral. In this paper, the author adduces an alternative proof for the Ptolemy's Theorem of cyclic quadrilaterals, involving a generalized corollary proved with respect to a cevian of an arbitrary Euclidean triangle covering the cases acute, obtuse, and right triangles, as well as for the converse of the Ptolemy's Theorem involving Mathematical Logic and different Mathematical Proofs.

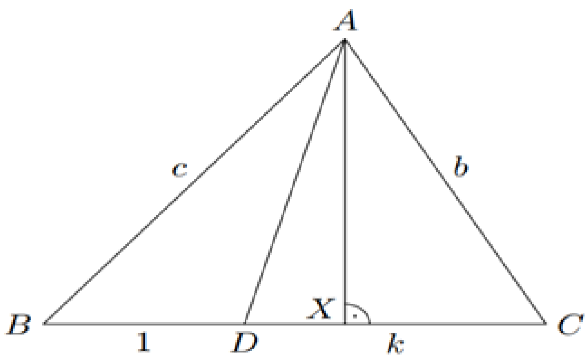
Main Results

Corollary 1

Let $ABC\Delta$ be an arbitrary plane triangle such that D be an arbitrary point on BC (an internal point), with $BC=a$, $AC=b$ and $AB=c$. If AD is a cevian such that $\frac{BD}{DC}=\frac{1}{k}$ for some $k>0$, then $AD^2=\frac{(1+k)(b^2+kc^2)-a^2k}{(k+1)^2}$

(Amarasinghe, 2011; Amarasinghe, 2012).

Figure 1.
An Euclidean triangle.



Proof of corollary

The proof of the corollary is a conditional proof under proof by cases. For the sake of simplicity (or without loss of generality), assume that $ABC\Delta$ is an acute angle triangle.

Case 1: Assume that AD is not perpendicular to BC.

Proof

Assume that AD is cevian such that $\frac{BD}{DC}=\frac{1}{k}$ Then draw the perpendicular AX to BC. Thus $DX\neq 0$. Using the Pythagoras Theorem respectively for $ABD\Delta$ (Obtuse Triangle), and $ABC\Delta$ (Acute Triangle), it follows that.

$c^2=AD^2-DX^2+(BD+DX)^2=AD^2+BD^2+2BD.DX$, and

$b^2=AD^2-DX^2+(DC-DX)^2=AD^2+DC^2-2DC.DX$

These results lead us to

$\frac{BD}{DC}=\frac{1}{k}=\frac{c^2-AD^2-BD^2}{AD^2+DC^2-b^2}$ since $k>0$ and $DX\neq 0$

Also, it is trivial to see that $BD=\frac{a}{(k+1)}$ and $DC=\frac{ka}{(k+1)}$. Thus, it follows that

$$\frac{1}{k}=\frac{c^2-AD^2-\left(\frac{a}{k+1}\right)^2}{AD^2+\left(\frac{ka}{k+1}\right)^2-b^2}$$

and after some elementary algebraic manipulation, this leads us to the desired result $AD^2=\frac{(1+k)\left(b^2+kc^2\right)-a^2k}{(k+1)^2}$.

Case 2: Assume that AD is perpendicular to BC. (Now X is coincided with D)

Proof

Then similarly, as before, using the Pythagoras Theorem, it follows $c^2=a^2+b^2-2a.DC$, as well as $b^2=a^2+c^2-2a.BD$.

Thus, it leads to $\frac{BD}{DC}=\frac{1}{k}=\frac{a^2+c^2-b^2}{a^2+b^2-c^2}$

. Thus $k=\frac{a^2+b^2-c^2}{a^2+c^2-b^2}$. Therefore

$k+1=\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)+1=\frac{2a^2}{a^2+c^2-b^2}$. Also,

it follows $BD = \frac{a^2 + c^2 - b^2}{2a}$.
Then observe that

$$AD^2 = c^2 - BD^2 = c^2 - \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 = \frac{4a^2c^2 - \left(a^2 + c^2 - b^2\right)^2}{4a^2}$$

Observe that

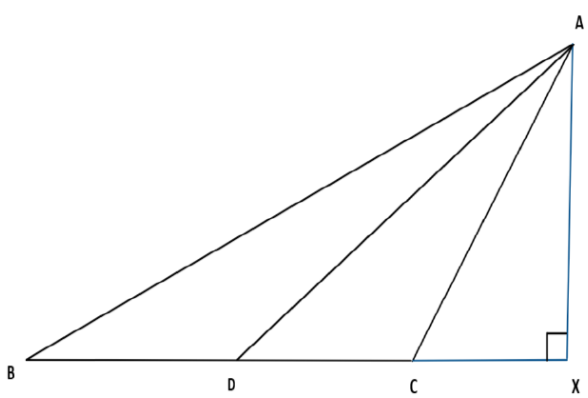
$$\frac{(1+k)(b^2+kc^2)\text{-}a^2k}{(k+1)^2} = \frac{\left(\frac{2a^2}{a^2+c^2-b^2}\right)\left(b^2+\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)c^2\right)-a^2\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}\right)}{\left(\frac{2a^2}{a^2+c^2-b^2}\right)^2}$$

$$= \frac{4a^2c^2\text{-}(a^2+c^2\text{-}b^2)^2}{4a^2}$$

$= AD^2$.

Hence it follows that in each case, $AD^2 = \frac{(1+k)(b^2+kc^2) - a^2k}{(k+1)^2}$.
Now it is not difficult to prove that, if $ABC\Delta$ is an obtuse triangle or a right-angled triangle, then also $AD^2 = \frac{(1+k)(b^2+kc^2) - a^2k}{(k+1)^2}$.
Assume that $ABC\Delta$ is an obtuse triangle. Without loss of generality, assume that the angle $A\hat{C}B$ is an obtuse angle.

Figure 2.
An obtuse Euclidean triangle.



Proof

Assume that AD is cevian such that $\frac{BD}{DC} = \frac{1}{k}$ for some $k > 0$. Then draw the perpendicular AX to extended BC. Thus $CX \neq 0$. Using the Pythagoras Theorem respectively for $ABD\Delta$ (Obtuse Triangle), and $ADC\Delta$ (Obtuse Triangle), it follows that,

$$c^2 = AD^2 - DX^2 + \left(BD + DX\right)^2 = AD^2 + BD^2 + 2BD.DX = AD^2 + BD^2 + 2BD.(DC + CX)$$

, and hence

$$c^2 = AD^2 + BD^2 + 2BD.(DC + CX) = AD^2 + BD^2 + 2BD.DC + 2BD.CX$$

$$.AD^2 = b^2 - CX^2 + \left(DC + CX\right)^2 = b^2 + DC^2 + 2DC.\,CX$$

These results lead us to

$$\frac{BD}{DC} = \frac{1}{k} = \frac{c^2 - AD^2 - BD^2 - 2BD.DC}{AD^2 - b^2 - DC^2}$$

since $k > 0$ and $CX \neq 0$. Also, it is trivial to see that $BD = \frac{a}{k+1}$ and $DC = \frac{ka}{k+1}$.

Thus, it follows that

$$\frac{BD}{DC} = \frac{1}{k} = \frac{c^2 - AD^2 - \left(\frac{a}{k+1}\right)^2 - 2\left(\frac{a}{k+1}\right).\left(\frac{ka}{k+1}\right)}{AD^2 - b^2 - \left(\frac{ka}{k+1}\right)^2}$$

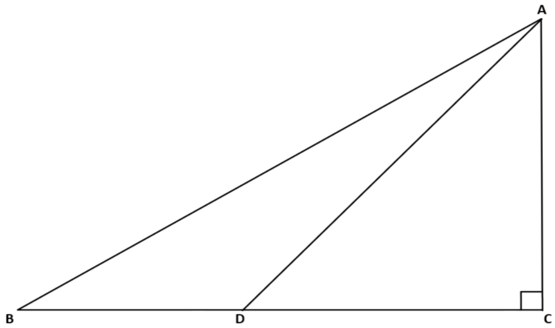
,after some elementary algebraic manipulation, this leads us to the desired result

$$AD^2 = \frac{(1+k)(b^2+kc^2) - a^2k}{(k+1)^2}$$

Assume that $ABC\Delta$ is a right-angled triangle**.** Without loss of generality, assume that the angle $A\hat{C}B$ is right-angle.

Figure 3

A right-angled Euclidean triangle.



Proof

Assume that AD is a cevian such that $\frac{BD}{DC} = \frac{1}{k}$ for some $k > 0$. Then AC is automatically perpendicular to BC. Using the Pythagoras Theorem respectively for $ADC\Delta$ (Right-triangle), it follows that,

$$c^2 = AD^2 = b^2 + DC^2 = b^2 + (BC - BD)^2 = b^2 + \left(a - \frac{a}{k+1}\right)^2$$

since $BD = \frac{a}{k+1}$, and hence this leads us to

$$AD^2 = b^2 + \frac{a^2k^2}{(k+1)^2} = \frac{k^2b^2 + kb^2 + kb^2 + b^2 + a^2k^2}{(k+1)^2} = \frac{k^2(a^2+b^2) + kb^2 + k(c^2 - a^2) + b^2}{(k+1)^2},$$

$$AD^2 = \frac{k^2c^2 + kb^2 + b^2 + kc^2 - a^2k}{(k+1)^2} = \frac{kc^2(k+1) + b^2(k+1) - a^2k}{(k+1)^2}.$$

Thus, this leads us to the required result

$$AD^2 = \frac{(k+1)(b^2 + kc^2) - a^2k}{(k+1)^2}.$$

Observe that now we have proved that for each Euclidean Triangle $ABC\Delta$, the above-mentioned result obtained for the length of the cevian AD, is true.

Theorem 1

Ptolemy's theorem

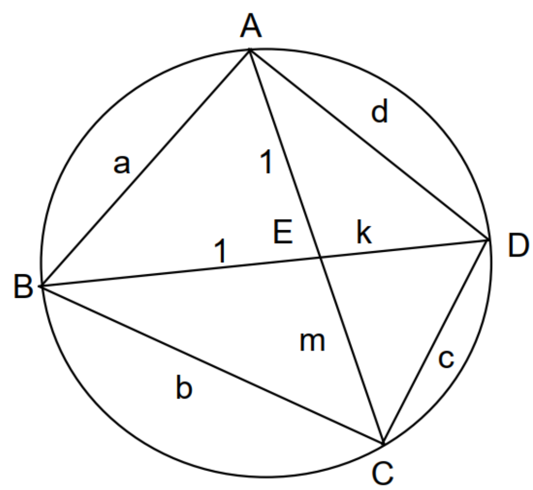
If $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals, then $AC.\,BD = AB.\,DC + AD.\,BC$. This is referred to as the Ptolemy's Theorem of Cyclic Quadrilaterals.

Novel proof

Assume that $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals. Suppose $AB = a$ $BC = b$, $CD = c$ and $AD = d$. Let E be the point of intersection of the diagonals AC and BD, and let $\frac{BE}{ED} = \frac{1}{k}$ and $\frac{AE}{EC} = \frac{1}{m}$ for some constants $k, m > 0$.

Figure 4.

A cyclic quadrilateral.



Since $\hat{BAE} = \hat{EDC}$, $\hat{ABE} = \hat{ECD}$ and, $ABE\Delta$ and $EDC\Delta$ are similar. Hence $\frac{BE}{EC} = \frac{a}{c}$.

Since $\hat{AD} = \hat{EBC}$, and $\hat{ADE} = \hat{ECB}$, $AED\Delta$ and

$BEC\Delta$ are similar. Hence $\frac{AE}{BE} = \frac{ED}{EC} = \frac{d}{b}$.

Thus $\left(\frac{BE}{EC}\right)\left(\frac{AE}{BE}\right) = \left(\frac{a}{c}\right)\left(\frac{d}{b}\right)$ which leads to $\frac{AE}{EC} = \frac{ad}{bc} = \frac{1}{m}$. Hence $m = \frac{bc}{ad}$.

Also, observe that $\frac{\left(\frac{BE}{EC}\right)}{\left(\frac{ED}{EC}\right)} = \frac{\left(\frac{a}{c}\right)}{\left(\frac{d}{b}\right)}$. Therefore, $\frac{BE}{ED} = \frac{ab}{cd} = \frac{1}{k}$. Hence $k = \frac{cd}{ab}$. Then by using the above corollary on cevians to $ABD\Delta$, we

yield $AE^2 = \frac{(1+k)(d^2+ka^2) - BD^2k}{(k+1)^2}$.

Similarly, by using the above corollary to $BCD\Delta$

, we yield $EC^2 = \frac{(1+k)(c^2+kb^2) - BD^2k}{(k+1)^2}$.

These two results lead to

$$\frac{AE^2}{EC^2} = \frac{(1+k)(d^2+ka^2) - BD^2k}{(1+k)(c^2+kb^2) - BD^2k} = \frac{1}{m^2}$$

By simplifying, this leads to

$$BD^2k(m^2-1) = (k+1)(m^2d^2 + m^2a^2k - c^2 - kb^2)$$

By substituting the above values for k and m, this leads to

$$BD^2\left(\frac{cd}{ab}\right)\left(\left(\frac{bc}{ad}\right)^2 - 1\right) = \left(\left(\frac{cd}{ab}\right)+1\right)\left(\left(\frac{bc}{ad}\right)^2 d^2 + \left(\frac{bc}{ad}\right)^2 a^2\left(\frac{cd}{ab}\right) - c^2 - \left(\frac{cd}{ab}\right)b^2\right)$$

By simplifying we have

$$BD^2(bc-ad)(bc+ad) = (ab+cd)(bd+ac)(bc-ad)$$

Case 1: Assume that $bc \neq ad$.
Then it easily follows

$$BD^2 = \frac{(ab+cd)(ac+bd)}{(ad+bc)}.$$

It is trivial to see that $AE = \frac{AC}{m+1}$ and

$EC = \frac{mAC}{m+1}$. Then observe that

$$AE^2 - EC^2 = \frac{(1+k)(d^2+ka^2) - BD^2k}{(k+1)^2} - \left[\frac{(1+k)(c^2+kb^2) - BD^2k}{(k+1)^2}\right] = \frac{k(a^2-b^2)+d^2-c^2}{k+1} =$$

$$\left(\frac{AC}{m+1}\right)^2 - \left(\frac{mAC}{m+1}\right)^2 = \frac{k(a^2-b^2)+d^2-c^2}{k+1} = AC^2\left(\frac{1-m^2}{(m+1)^2}\right) = AC^2\left(\frac{1-m}{1+m}\right).$$

By substituting the above values for k and m, this leads to

$$AC^2\left(\frac{1-\left(\frac{bc}{ad}\right)}{\left(1+\frac{bc}{ad}\right)}\right) = \frac{\left(\frac{cd}{ab}\right)(a^2-b^2)+d^2-c^2}{\left(\frac{cd}{ab}\right)+1}$$

. Hence $AC^2\frac{(ad-bc)}{ad+bc} = \frac{(ac+bd)(ad-bc)}{ab+cd}$

Since by our assumption, $bc \neq ad$, it easily

follows that $AC^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}$.

Case 2: Assume that $bc = ad$.

Then since $m = \frac{bc}{ad}$, it follows $m = 1$. That is, then E is the midpoint of AC.

Then by using the Apollonius Theorem for the $ADC\Delta$, it follows that $2AE^2 + 2ED^2 = d^2 + c^2$ Observe that by the above-mentioned similar triangles

$$ED = EC\left(\frac{d}{b}\right), \text{ and } EC = \frac{mAC}{m+1},$$

it follows that

$$ED=\left(\frac{mAC}{m+1}\right)\left(\frac{d}{b}\right)=\left(\frac{\left(\frac{bc}{ad}\right)AC}{\left(\frac{bc}{ad}\right)+1}\right)\left(\frac{d}{b}\right)=\frac{AC.\,cd}{bc+ad}$$

Moreover,

$$AE=\frac{AC}{m+1}=\frac{AC}{\left(\frac{bc}{ad}\right)+1}=\frac{ACad}{ad+bc}$$

Thus, by the above Apollonius Theorem, it follows that

$$2\left(\frac{ACad}{ad+bc}\right)^2+2\left(\frac{AC.\,cd}{bc+ad}\right)^2=d^2+c^2$$

By simplifying this further, since $bc=ad$, and rearranging the terms, we yield to the desired result, $AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}$.

Observe that $EC=BE\left(\frac{c}{a}\right)$. Since $BE=\frac{BD}{k+1}$, it follows $EC=\left(\frac{BD}{k+1}\right)\left(\frac{c}{a}\right)$

Then from the above proved relation, we have

$$EC^2=\frac{(1+k)(c^2+kb^2)-BD^2k}{(k+1)^2}=\left(\frac{BDc}{a(k+1)}\right)^2$$

Substituting for k, we have

$$\frac{BD^2c^2}{a^2\left(\frac{cd}{ab}+1\right)^2}=\frac{\left(1+\frac{cd}{ab}\right)\left(c^2+\left(\frac{cd}{ab}\right)b^2\right)-BD^2\left(\frac{cd}{ab}\right)}{\left(\frac{cd}{ab}+1\right)^2}$$

which leads to $BD^2=\frac{(ab+cd)(ac+bd)}{(ad+bc)}$. That is in each case

$$AC^2=\frac{(ad+bc)(ac+bd)}{ab+cd}$$

and $BD^2=\frac{(ab+cd)(ac+bd)}{(ad+bc)}$.

Hence, we yield

$$AC^2.\,BD^2=\frac{(ad+bc)(ac+bd)}{ab+cd}\times\frac{(ab+cd)(ac+bd)}{(ad+bc)}=(ac+bd)^2$$

Hence, it easily follows $AC.\,BD=AB.\,DC+AD.\,BC$ which is the Ptolemy's Theorem of Cyclic Quadrilaterals. This completes the proposed alternative proof of Ptolemy's Theorem (Alsina & Nelson, 2007; Amarasinghe, 2023).

Remark 1

It also follows that $\frac{AC}{BD}=\frac{ad+bc}{ab+cd}$.

Corollary 2

Assume that $ABCD$ is a cyclic quadrilateral such that AC and BD are its diagonals, and $AB=a$ $BC=b$, $CD=c$ and $AD=d$. Then the intersection point E of the diagonals is the midpoint of AC if and only if $bc=ad$.

Proof of corollary 2

Proof is trivial under the above case 2, if $m=1$.

The converse of the Ptolemy's theorem (converse of Theorem 1)

Let A,B,C and D be four arbitrary points in