2024 SICASH
SLIIT INTERNATIONAL CONFERENCE ON ADVANCEMENTS IN SCIENCES & HUMANITIES

The Geometry Involving a Right-Cevian Triangle

G.W. Indika Shameera Amarasinghe
Sri Lanka Institute of Information Technology, Sri Lanka
Email address of the corresponding author - *indika.a@sliit.lk

Abstract

This paper presents some significant theorems and corollaries prevailed, involving the cevian triangles, in particular, on right cevian triangles. Some useful corollaries are presented including a main theorem on right cevian triangles with detailed proofs, involving the ratios generated by the end points of internal cevians drawn to the side lengths of a triangle, adducing more generalized formulas for the different cevians of triangles around the right cevian triangle to fulfil the existing significant research gap in cevian geometry, particularly on cevian triangles. This paper presents pure geometric proofs for the theorems and corollaries, without using trigonometry, coordinate geometry, complex numbers, vectors or any other non-geometric approaches.

Keywords: Cevian; Cevian Triangles; Mathematical Logic; Inequalities; Cubic Equations

Introduction

The geometry developed by Giovanni Ceva who was an Italian eminent Mathematician, on the cevians of triangles, is usually called Cevian Geometry. A cevian can be split to two categories, namely, internal cevian, and the external cevian, from which the most prominent Mathematical Theorems, Lemmas and Corollaries were proved. He has published famous theorems, including one known as Ceva's Theorem on concurrent cevians of a triangle. A cevian is a line segment that extends from one vertex of a triangle to the opposite side; this can be internal (internal cevian) if the cevian's end point is within the opposite side, or can be external (external cevian), if the cevian's end point is on the opposite extended side length of the triangle. (Hajja, 2006; Amarasinghe, 2011) The length AD in Figure 1 is an example for a cevian (internal cevian). The triangle generated by the end points of the three concurrent cevians of a triangle, is called a cevian triangle. In this paper, the author explores on the geometry of cevian triangles; in particular, the right-cevian triangles.

The author himself has published a new theorem on an arbitrary right-cevian triangle (Amarasinghe, 2011) using purely Euclidean Geometry (without using trigonometry, vector algebra, complex numbers or any other techniques), nevertheless the theorem was published based on a more problem-solving approach rather than an explicit research approach due to the readability of the relevant journal. In this paper, the author revamps the right-cevian triangle theorem and its proof within a more research-focused environment, consequently deriving a significant number of useful theorems and corollaries on right-cevian triangles, along with the proofs of their converses and logically equivalent corollaries, including some inequalities. Moreover, some important corollaries which involve a cevian, but may not include the cevian triangles, are also presented.

https://doi.org/10.54389/CXCC3246

Main Results

Lemma 1

Let $ABC\Delta$ be an arbitrary plane triangle such that D be an arbitrary point on BC (an internal point), with $BC = a$, $AC = b$ and $AB = c$. For each constant $k > 0$, If AD is a cevian such that $\frac{BD}{DC} = \frac{1}{k}$, then the length of the cevian AD is given by

$$AD^2 = \frac{(1+k)\left(b^2 + kc^2\right) - a^2 k}{(k+1)^2}.$$

(Amarasinghe, 2010a; Amarasinghe, 2010b; Amarasinghe, 2011)

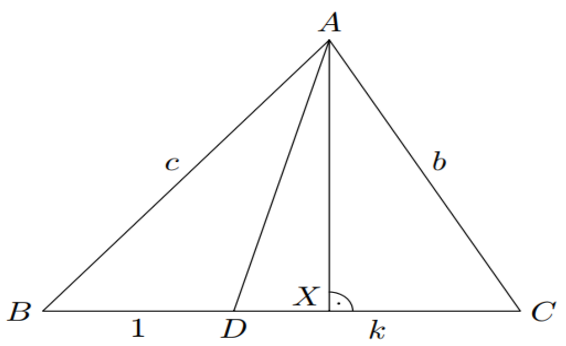

Figure 1. *A Euclidean Triangle*

Remark 1. Observe that the same formula for the cevian AD can be proved, if $ABC\Delta$ is an obtuse triangle or $ABC\Delta$ is a right triangle, and if AD is perpendicular to BC (Amarasinghe, 2023).

Theorem 1

Let $ABC\Delta$ be a Euclidean plane triangle such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian center O. If $DEF\Delta$ is a right-triangle, such that $E\hat{D}F$ is a right-angle, then DE and DF are the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively.

(Amarasinghe, 2011)

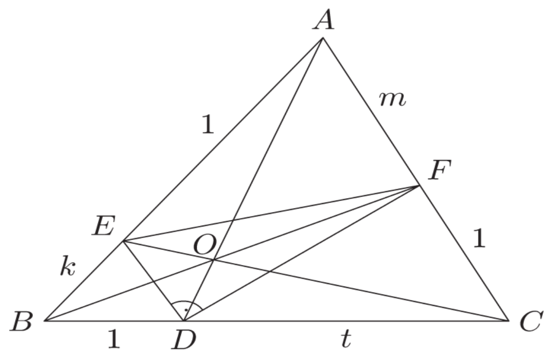

Figure 2. A triangle with an inscribed Right-cevian triangle

Proof (Theorem 1)

Since D, E, F are arbitrary points (as mentioned in the above figure), let $\frac{BD}{DC} = \frac{1}{t}$, $\frac{AE}{BE} = \frac{1}{k}$ and $\frac{CF}{AF} = \frac{1}{m}$ for arbitrary positive constants k, t, m. Let $BC = a$, $AC = b$ and $AB = c$.

Since AD, BF and CE are concurrent at the cevian center O, by using the Ceva's Theorem, it follows that $\frac{1}{t} \cdot \frac{1}{m} \cdot \frac{1}{k} = 1$. Thus, $m = \frac{1}{kt}$. (Karapetoff, 1929; Amarasinghe, 2011).

Using lemma 1 for the cevian AD in $ABC\Delta$, we obtain

$$AD^2 = \frac{(1+t)\left(b^2 + tc^2\right) - a^2 t}{(t+1)^2}.$$

Using lemma 1 for the cevian DE in $ADB\Delta$, we obtain

$$DE^2 = \frac{(1+k)\left(BD^2 + kAD^2\right) - c^2 k}{(k+1)^2}.$$

Since $BD = \frac{a}{1+t}$, it follows that

$$DE^2=\frac{(1+k)\left(\left(\frac{a}{1+t}\right)^2+k\left(\frac{(1+t)(b^2+tc^2)-a^2t}{(t+1)^2}\right)\right)-c^2k}{(k+1)^2}$$
.

Further simplification leads us to

$$DE^2=\frac{a^2(1-kt)+k(t+1)(b^2+tc^2)}{(t+1)^2(k+1)}-\frac{c^2k}{(k+1)^2}.$$

Remark 2. Observe that the lengths of the cevians AD and DE obtained above are independent of the concurrence of three cevians AD, BE and CF.

Using lemma 1 for the cevian DF in $ADC\Delta$, we yield $DF^2=\frac{(1+m)(AD^2+mDC^2)-b^2m}{(m+1)^2}$.

Since $DC=\frac{at}{1+t}$ and $m=\frac{1}{kt}$, it follows that

$$DF^2=\frac{\left(1+\frac{1}{kt}\right)\left(\frac{(1+t)(b^2+tc^2)-a^2t}{(t+1)^2}+\left(\frac{1}{kt}\right)\left(\frac{at}{1+t}\right)^2\right)-b^2\left(\frac{1}{kt}\right)}{\left(\frac{1}{kt}+1\right)^2}$$

. Further simplification leads us to

$$DF^2=\frac{\left[a^2t(1-k)+k(t+1)(b^2+tc^2)\right]t}{(t+1)^2(kt+1)}-\frac{b^2kt}{(kt+1)^2}$$

.

Observe that using lemma 1 for the cevian CE in $ABC\Delta$, we yield $CE^2=\frac{(1+k)(a^2+kb^2)-c^2t}{(k+1)^2}$.

Using lemma 1 for the cevian EF in $AEC\Delta$, we yield $EF^2=\frac{(1+m)(AE^2+mCE^2)-b^2m}{(m+1)^2}$

. Since $AE=\frac{c}{1+t}$ and $m=\frac{1}{kt}$, it follows that

$$EF^2=\frac{\left(1+\frac{1}{kt}\right)\left(\left(\frac{c}{1+t}\right)^2+\left(\frac{1}{kt}\right)\left(\frac{(1+k)(a^2+kb^2)-c^2t}{(k+1)^2}\right)\right)-b^2\left(\frac{1}{kt}\right)}{\left(\frac{1}{kt}+1\right)^2}$$

. Further simplification leads us to

$$EF^2=\frac{c^2k(t-1)+(k+1)(a^2+kb^2)}{(k+1)^2(kt+1)}-\frac{b^2kt}{(kt+1)^2}.$$

Now assume that $DEF\Delta$ is a right triangle. Then by using the Pythagoras Theorem, we obtain $EF^2=DE^2+DF^2$. Hence it follows that

$$\frac{c^2k(t-1)+(k+1)(a^2+kb^2)}{(k+1)^2(kt+1)}-\frac{b^2kt}{(kt+1)^2}$$
$$=\frac{a^2(1-kt)+k(t+1)(b^2+tc^2)}{(t+1)^2(k+1)}-\frac{c^2k}{(k+1)^2}+$$
$$\frac{\left[a^2t(1-k)+k(t+1)(b^2+tc^2)\right]t}{(t+1)^2(kt+1)}-\frac{b^2kt}{(kt+1)^2}.$$

A careful simplification gradually leads us to $k^2\left((t+1)(b^2+tc^2)-a^2t\right)-a^2=0$. Since $k>0$, it follows that $k=\frac{a}{\sqrt{(1+t)(b^2+tc^2)-a^2t}}$. Since $AD^2>0$, it follows that $(1+t)(b^2+tc^2)-a^2t>0.$ Thus k does exist such that $k\in\mathbb{R}$

Now observe that

$$\frac{BE}{AE}=k=\frac{a}{t+1}.\frac{t+1}{\sqrt{(1+t)(b^2+tc^2)-a^2t}}=\frac{BD}{AD}$$
.

Hence by the converse of angle bisector theorem, it follows that DE is the internal angle bisector of the angle $A\widehat{D}B$.

Also observe that

$$\frac{AF}{FC}=m=\frac{1}{kt}=\frac{\sqrt{(1+t)(b^2+tc^2)-a^2t}}{at}=\frac{t+1}{at}.\frac{\sqrt{(1+t)(b^2+tc^2)-a^2t}}{t+1}=\frac{AD}{DC}$$

Observe that since $(1+t)(b^2+tc^2)-a^2t>0$, m does exist such that $m\in\mathbb{R}$ (Amarasinghe, 2012).

Hence by the converse of angle bisector theorem, it follows that DF is the internal angle bisector of the angle $A\hat{D}C$. This completes the proof of Theorem 1.

Converse of Theorem 1

Let $ABC\Delta$ be a Euclidean plane triangle such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian centre O. If DE and DF are the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively, then $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle.

Proof. (Converse of Theorem 1)

Assume that DE and DF are the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively. Then $E\hat{D}B = A\hat{D}E$ and $A\hat{D}F = F\hat{D}C$. Since $E\hat{D}B + A\hat{D}E + A\hat{D}F + F\hat{D}C = 180^0$, it easily follows that

$A\hat{D}E + A\hat{D}F = 90^0$. Thus $E\hat{D}F = 90^0$. Hence $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is right-angle.

Theorem 2 (Logically Equivalent to Theorem 1)

Let $ABC\Delta$ be a Euclidean plane triangle such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian centre O. If at least one of the lengths DE and DF are not internal angle bisectors of angles $A\hat{D}B$ and $A\hat{D}C$ respectively, then the cevian triangle $DEF\Delta$ is not a right triangle such that $E\hat{D}F$ is a right-angle.

Proof. The proof is trivial. Use the contrapositive of the conditional statement involved in Theorem 1.

Theorem 3

Let $ABC\Delta$ be a Euclidean plane triangle be such that AD is an arbitrary internal cevian AD, and E and F are arbitrary points on AB and AC respectively. If DE and DF are the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively, then $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle, and the cevians AD, BF and CE are concurrent at a point O.

Proof. Assume that DE and DF are the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively. Then from the converse of the Theorem 1, it easily follows that $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is right-angle. Observe that by using the angle bisector Theorem, it follows that

$\frac{AD}{DB}=\frac{AE}{BE}$ and $\frac{AD}{DC}=\frac{AF}{FC}$. Observe that that $\frac{BD}{DC}.\frac{CF}{AF}.\frac{AE}{BE}=\frac{BD}{DC}.\frac{DC}{AD}.\frac{AD}{BD}=1$. Hence by the converse of the Ceva's Theorem, it follows that the cevians AD, BF and CE are concurrent at a point O.

Corollary 1. (Under Theorem 1)

Let $ABC\Delta$ be a Euclidean plane triangle be such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian center O. If $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle and $\frac{BE}{AE}=\frac{DC}{BD}=\frac{AF}{FC}$, then the $ABC\Delta$ is also a right triangle.

Proof. (Corollary 1)

Assume that $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle and $\frac{BE}{AE}=\frac{DC}{BD}=\frac{AF}{FC}$. Then as in figure 2, $\frac{BE}{AE}=k=\frac{DC}{BD}=t=\frac{AF}{FC}=m$. Ceva's Theorem leads us to $\frac{1}{t}.\frac{1}{m}.\frac{1}{k}=1$. Thus $\frac{1}{t^3}=1$. Since $t>0$, $t=1$. Thus $t=k=m=1$. Thus, it follows that D, E and F are mid points of BC, AB and AC. By using the mid-point Theorem in Euclidean Geometry, it follows that DE is parallel with AC, and DF is parallel with AB. Hence, $AEDF$ is a parallelogram. Thus, $E\hat{D}F = B\hat{A}C$. Since $E\hat{D}F$ is a right-angle, $90^0 = E\hat{D}F = B\hat{A}C$. Hence, $ABC\Delta$ is a right triangle.

Corollary 2 (Under the Theorem 1)

Let $ABC\Delta$ be a Euclidean plane triangle such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian center O. If $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle, then $EF^2 = AD.BC - AE.BE - AF.FC$.

Proof. (Corollary 2)

Assume that $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is right-angle. Then by Theorem 1, we have DE and DF are the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively. Thus, by a well-known corollary related to the lengths of angle bisectors, we yield $DE^2 = AD.BD - AE.BE$ and $DF^2 = AD.DC - AF.FC$. Since $DEF\Delta$ is a right-triangle, it follows that

$DE^2 + DF^2 = AD(BD + DC) - AE.BE - AF.FC = EF^2$. That is, we have led to the required result $EF^2 = AD.BC - AE.BE - AF.FC$ (Amarasinghe, 2011, Amarasinghe 2012).

Corollary 3 (Under Corollary 2)

Let $ABC\Delta$ be an Euclidean plane triangle such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian center O. If $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle with $EF = AD = AE = AF$, then $a = BC = AD + BE + FC$.

Proof. (Corollary 3)

Assume that $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle with $EF = AD = AE = AF$. Then by corollary 2, we obtain $EF^2 = AD.BC - AE.BE - AF.FC$. Thus, $AD^2 = AD.BC - AD.BE - AD.FC$. Hence, it follows that $a = BC = AD + BE + FC$.

Corollary 4 (An Inequality)

Let $ABC\Delta$ be a Euclidean plane triangle such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian center O. If $DEF\Delta$ is a right-triangle, such that $E\hat{D}F$ is a right-angle, then $EF < \sqrt{a(a+b+c)}$.

Proof. (Corollary 4)

Assume that $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is right-angle. Then by corollary 2, we obtain $EF^2 = AD.BC - AE.BE - AF.FC$. Observe that $EF^2 < AD.BC$. Also, observe that using the triangle inequality for $ADB\Delta$ and $ADC\Delta$ triangles, we yield $AD < c + BD$ and $AD < b + DC$ respectively. Adding them together leads to $2AD < c + b + BD + DC = c + b + a$. Hence,

$AD < \frac{a+b+c}{2} < a+b+c$. Hence, it follows that $EF^2 < AD.BC < a\left(a+b+c\right)$. Thus, it follows that $EF < \sqrt{a\left(a+b+c\right)}$.

Extended Corollary 4 (An Inequality)

Let EF, QR and TU be the hypotenuses of the three arbitrary right-cevian triangles constructed with their right-angle vertexes located on BC, CA and AB respectively.

Then $EF^2 + QR^2 + TU^2 < \left(a+b+c\right)^2$.

Proof. (Extended Corollary 4)

By corollary 4, it follows that $EF^2 < a\left(a+b+c\right)$.

Hence, we can deduce that $QR^2 < b\left(a+b+c\right)$ and $TU^2 < c\left(a+b+c\right)$.

Thus, it follows that

$$EF^2 + QR^2 + TU^2 < a(a+b+c) + b(a+b+c) + c(a+b+c) = (a+b+c)(a+b+c) = (a+b+c)^2.$$

Now it is interesting to analyze the case of the geometry around the right cevian triangle $DEF\Delta$, if the main triangle $ABC\Delta$ is an equilateral triangle.

Corollary 5 (under Theorem 1)

Let $ABC\Delta$ be an equilateral Euclidean plane triangle be such that the internal arbitrary cevians AD, BF and CE are concurrent at the cevian center O and $\frac{BE}{AE} = k$ and $\frac{DC}{BD} = t$. If $DEF\Delta$ is a right-cevian triangle such that $E\widehat{D}F$ is a right-angle, then $k = \frac{1}{\sqrt{t^2+t+1}}$.

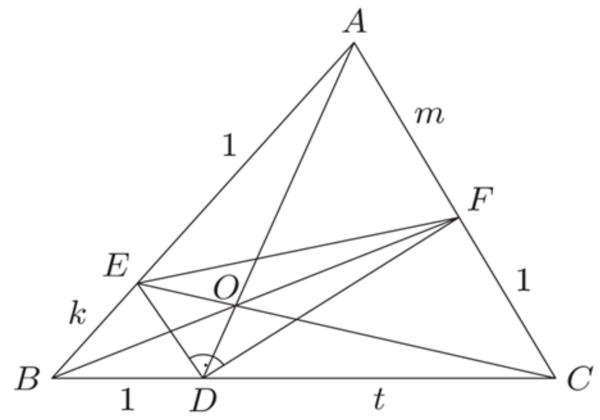

Figure 3. An equilateral triangle with an inscribed right-cevian triangle

Proof. (Corollary 5)

Proof is trivial due to the proof of Theorem 1. Assume that $DEF\Delta$ is a right-cevian triangle such that $E\widehat{D}F$ is right-angle. The by the proof of theorem 1, it follows that

$$k = \frac{a}{\sqrt{(1+t)\left(b^2+tc^2\right)-a^2t}} = \frac{a}{\sqrt{(1+t)\left(a^2+ta^2\right)-a^2t}} = \frac{1}{\sqrt{t^2+t+1}}.$$

This completes the proof.

Observe that the following important remarks can be established under the corollary 5.

Remark 3 Then it also trivially follows that since $k = \frac{1}{\sqrt{t^2+t+1}}$ implies that k is only dependent on t and not dependent of the side lengths of the main triangle $ABC\Delta$.

Remark 4. Observe that the converse of the corollary 5 trivially holds from the converse of the Theorem 1. (Since the converse holds for an arbitrary triangle $ABC\Delta$)

Remark 5 Rearranging the terms in corollary 5, reduces to $t^2+t+1-\frac{1}{k^2}=0$. Since the discriminant of this quadratic is non-negative, it follows that $1-4\left(1-\frac{1}{k^2}\right)\geq 0$. Since $k>0$, it follows that $k\leq\frac{2}{\sqrt{3}}$. Thus $\frac{BE}{AE}\leq\frac{2}{\sqrt{3}}$. Also, it is not difficult to prove that the maximum value of k is $\frac{2}{\sqrt{3}}$.

Now it is interesting to analyse what happens, if $k=\frac{1}{\sqrt{1+2t}}$ and $DEF\Delta$ is **not** a cevian triangle, also given that $ABC\Delta$ is equilateral. The following corollary reveals about it. Please recall that now we assume that the only cevian of the main triangle $ABC\Delta$ is AD.

Corollary 6.

Let $ABC\Delta$ be an equilateral Euclidean plane triangle such that AD is an internal cevian. Let E be a point on AB such that DE is a cevian of $ABD\Delta$, and let $\frac{BE}{AE}=k$ and $\frac{DC}{BD}=t$. If $k=\frac{1}{\sqrt{1+2t}}$ and DE is the internal angle bisector of the angle $A\hat{D}B$, then AD is a median of the equilateral triangle $ABC\Delta$.

Proof. (Corollary 6)

Assume that $k=\frac{1}{\sqrt{1+2t}}$ **and** DE is the internal angle bisector of the angle $E\hat{D}F$. Observe that by the angle bisector theorem, it follows that $\frac{AD}{BD}=\frac{AE}{BE}$. Then it follows that

$$\frac{\frac{\sqrt{(1+t)\left(b^2+tc^2\right)-a^2t}}{t+1}}{\frac{a}{t+1}}=\frac{1}{k}=\sqrt{1+2t}$$

. Since $ABC\Delta$ is an equilateral triangle, it follows that

$$\frac{\frac{\sqrt{(1+t)\left(a^2+ta^2\right)-a^2t}}{t+1}}{\frac{a}{t+1}}=\sqrt{(t+1)^2-t}=\sqrt{1+2t}$$

.

That is $(t+1)^2-t=1+2t$. This leads us to $t(t-1)=0$. Thus $t=0$ or $t=1$. Since $t>0$, it follows that $t=1$. That is D is the mid-point of BC. Thus AD is a median of $ABC\Delta$. This completes the proof of corollary 6.

Remark 6 According to the conclusion of the corollary 6, since $ABC\Delta$ is an equilateral triangle, AD trivially becomes a perpendicular drawn to AB and also $k=\frac{1}{\sqrt{3}}$.

The following corollary is immediately followed by corollary 6.

Corollary 7 (Logically equivalent to corollary 6)

Let $ABC\Delta$ be an equilateral Euclidean plane triangle be such that AD is an internal cevian. Let E be a point on AB such that DE is a cevian of $ABD\Delta$, and let $\frac{BE}{AE}=k$ and $\frac{DC}{BD}=t$.

If AD is not a median of the equilateral triangle $ABC\Delta$, then $k\neq\frac{1}{\sqrt{1+2t}}$ or DE is **not** the internal angle bisector of the angle $A\hat{D}B$.

Proof. (Corollary 7)

The proof is trivially followed by the contrapositive of the conditional statement of corollary 6.

Corollary 8 (An inequality)

Let $ABC\Delta$ be an equilateral Euclidean plane triangle be such that the internal arbitrary cevians

AD, BF and CE are concurrent at the cevian center O. If $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle, and $\frac{BE}{AE} = k = \frac{DC}{BD} = t$, then $\frac{3}{5} \leq \frac{BE}{AE} = \frac{DC}{BD} \leq \frac{7}{10}$ and D, E and F are fixed points of the equal side lengths BC, AB and AC respectively.

Proof. (Corollary 8) Assume that $DEF\Delta$ is a right-triangle such that $E\hat{D}F$ is a right-angle**, and** $\frac{BE}{AE} = k = \frac{DC}{BD} = t$. Then by corollary 5, it follows that $k = \frac{1}{\sqrt{t^2+t+1}} = t$. Thus $t^4 + t^3 + t^2 - 1 = 0$. Thus, $(t+1)(t^3+t-1) = 0$. Since $t > 0$, $t^3 + t - 1 = 0$. Put $t^3 + t - 1 = f(t)$. Observe that it is not difficult to prove that $f(t)$ is continuous on $\left[\frac{6}{10}, \frac{7}{10}\right] = [0.6, 0.7]$ by using the $(\varepsilon - \delta)$ definition of continuity on an interval. Observe that $f\left(\frac{6}{10}\right) = \left[\frac{6}{10}\right]^3 + \frac{6}{10} - 1 = \frac{-184}{1000} < 0$, and $f\left(\frac{7}{10}\right) = \left[\frac{7}{10}\right]^3 + \frac{7}{10} - 1 == \frac{43}{1000} > 0$. Thus, by the Intermediate value theorem, there exists a real root $t_0 \in \left[\frac{6}{10}, \frac{7}{10}\right]$ such that $f(t_0) = 0$.

By using Cardona's method of solving cubic equations of the standard form $at^3 + bt^2 + ct + d = 0$, the standard discriminant Δ of the cubic is given by $\Delta = 18abcd + b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2$. If the cubic is depressed of the form $t^3 + pt + q = 0$ then it is trivial to see that the discriminant reduces to $\Delta = -4p^3 - 27q^2$. Observe that since $t^3 + t - 1 = 0$, it follows $\Delta = -31 < 0$. Hence it follows that the cubic has exactly one real root and two complex roots. That is $t^3 + t - 1 = 0$ has only one real root $t \in \left[\frac{6}{10}, \frac{7}{10}\right]$. (Note that we can exactly calculate the exact real root by proceeding the Cardona's method). That is $\frac{3}{5} \leq \frac{BE}{AE} = \frac{DC}{BD} \leq \frac{7}{10}$. Since the equation has a fixed real root, it follows that both k, t are fixed. Thus, the points D, E are fixed points of the equal side lengths BC and AB respectively. Since $m = \frac{1}{kt} = \frac{1}{t^2}$ (see the proof of theorem 1), it follows that F is also a fixed point on AC. This completes the proof of Corollary 8.

Conclusions

The readers are invited to figure out the fact that whenever the cevians are concurrent at a point O (cevian center), the ratios generated by the end points of cevians on the side lengths of a triangle, conform a mutual correlation. In particular, no matter what the main triangle, we choose ($ABC\Delta$), no matter what the right-cevian triangle is generated inside the $ABC\Delta$, it follows that the side lengths DE and DF become the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively from which it follows the length of the hypotenuse of the right-cevian triangle $DEF\Delta$, and thereby led to a useful inequality in terms of the side lengths of $ABC\Delta$. Observe that it is intricate to intuitively deduce or speculate (even if a graphical approach is used) that DE and DF become the internal angle bisectors of the angles $A\hat{D}B$ and $A\hat{D}C$ respectively as soon as we depict the right-cevian triangle in a figure at the first glance, and only the proof can convey it. Moreover, the converse of the Theorem is proved, and a useful corollary which is almost equivalent to the converse of the theorem, is also presented and the contrapositive of a conditional statement is used several times to prove some new corollaries which are significant, with ease. The corollary 6 only includes a cevian, but not a cevian triangle, but eventually led to a result which is more interesting. Such proofs convey the idea that Pure Mathematics is magical and gorgeous. Eventually, the subsidy of solving the cubic equations using the Cardona method, and the Intermediate value theorem in real analysis, played a major role in proving the corollary 8, showcasing the fact that Advanced Euclidean Geometry often needs the subsidy of other branches of Pure Mathematics to successfully prove significant theorems and corollaries.

References

Amarasinghe, I. (2010a). Advanced plane geometry research 1. *Proceedings of the 66th Annual Sessions of the Sri Lanka Association for the Advancement of Science (SLAAS)*, *66*, 77.

Amarasinghe, I. (2010b). Advanced plane geometry research 3: Alternative proofs for the standard theorems in plane geometry. *Proceedings of the 66th Annual Sessions of the Sri Lanka Association for the Advancement of Science (SLAAS)*, *66*, 78.

Amarasinghe, I. (2011). A new theorem on any right-angled cevian triangle. *Journal of the World Federation of National Mathematics Competitions (JWFNMC)*, *24*(2), 29-37.

Amarasinghe, I. (2012). On the standard lengths of angle bisectors and the angle bisector theorem. *Global Journal of Advanced Research on Classical and Modern Geometries (GJARCMG)*, 15-27.

Amarasinghe, I. (2023). An Alternative Proof of Ptolemy's Theorem and Its Variations, *SLIIT Journal of Humanities and Sciences (SJHS)*, *4*(1), 1-11.

Hajja, M. (2006). The arbitrariness of the cevian triangle. *The American Mathematical Monthly*, *113*(5), 443-447.

Karapetoff, V. (1929). Some properties of correlative vertex lines in a plane triangle. *The American Mathematical Monthly*, *36*(9), 476-479.