

# Influence of topology in the evolution of coordination in complex networks under information diffusion constraints

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Received 11 September 2013 / Received in final form 29 October 2013

Published online 7 January 2014 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2014

**Abstract.** In this paper, we study the influence of the topological structure of social systems on the evolution of coordination in them. We simulate a coordination game (“Stag-hunt”) on four well-known classes of complex networks commonly used to model social systems, namely scale-free, small-world, random and hierarchical-modular, as well as on the well-mixed model. Our particular focus is on understanding the impact of information diffusion on coordination, and how this impact varies according to the topology of the social system. We demonstrate that while time-lags and noise in the information about relative payoffs affect the emergence of coordination in all social systems, some topologies are markedly more resilient than others to these effects. We also show that, while non-coordination may be a better strategy in a society where people do not have information about the payoffs of others, coordination will quickly emerge as the better strategy when people get this information about others, even with noise and time lags. Societies with the so-called small-world structure are most conducive to the emergence of coordination, despite limitations in information propagation, while societies with scale-free topologies are most sensitive to noise and time-lags in information diffusion. Surprisingly, in all topologies, it is not the highest connected people (hubs), but the slightly less connected people (provincial hubs) who first adopt coordination. Our findings confirm that the evolution of coordination in social systems depends heavily on the underlying social network structure.

## 1 Introduction

Studying the behavioural or evolutionary dynamics of a population has played a central role in our understanding of emergent phenomena. Such studies have shed light on biological systems from cells [1] to species [2], on international politics [3] and even on the firing of populations of neurons as we try to understand the internal states of another person’s mind [4]. Modern approaches in this field began with the work of Von Neumann and Morgenstern [5] in 1944 and were taken up by John Nash who developed one of the most influential ideas in game theory, that of the Nash equilibrium [6]. The Nash equilibrium states that in an incentivised situation between two or more strategic players of a “game”, at least one choice of strategy can be found whereby no other player can achieve a better outcome by unilaterally changing their strategy. This particular notion of equilibrium and its subsequent extensions have profoundly influenced our understanding of game theory as well as strategic interactions more broadly.

However, it quickly became apparent that the notion of a Nash equilibrium had a significant flaw: the cooperation that had been observed between players in behavioural experiments and we observe in everyday life was not an “Evolutionary Stable Strategy” (ESS) [7], i.e. the

theoretical result of Nash did not reflect empirical observations. While successful methods to address this shortcoming have recently been explored using both finite and infinite population sizes [8], the disconnect between theory and observation was first addressed when the study of spatially extended games was introduced [9–12]. In these games, interactions between agents were based on their spatial relationship rather than a so-called “well-mixed” population. In these studies, interacting strategic agents played games in a two-dimensional space, typically on a lattice, and only nearest neighbours on the lattice interacted with one another, a more physically realistic model than that of a well-mixed population where every agent can potentially interact with every other agent. In this scenario it was shown that cooperation was a stable strategy and a portion of cooperating agents were able to persist indefinitely in the system, thereby lining up theory with observation.

These results were then extended to other topological spaces by using different network topologies, generalising the idea of a rigid lattice to that of stochastic connections between agents. In these models the relationships between individuals are not described by spatial connectedness but by more abstract connections such as the role of a species in a food web [13] or people connected via a social network [14]. Such generalisations have significantly

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broadened and deepened the phenomena that game theory has been able to explore.

Networked game theory has progressed significantly since the introduction of the so-called small-world and scale-free topologies into the more general field of network theory more than a decade ago [15–18]. One of the more significant results to come from this work was a result by Ohtsuki et al. [19] where a general rule for when cooperation is favoured over non-cooperation for different network topologies was developed. Labelling benefits of cooperation as  $b$  and the costs of cooperation as  $c$ , and given the average  $\bar{k}$  number of connections an agent has to other agents, cooperators are favoured when  $b/c > \bar{k}$ . This very general result holds for a number of broad classes of networks and works as a very good rule of thumb. However, as pointed out in Ohtsuki et al.’s original paper, this analysis is a poor estimate in the case of scale-free networks and further study is needed in order to understand what role is played by scale-free networks (and other well known network topologies) that leads to such special circumstances.

With this background in mind, this research compares the evolutionary game dynamics over four different network topologies commonly used to model social systems, namely the scale-free, small-world, hierarchical-modular, and Erdős-Rényi random topologies [20]. We also consider regular lattices (where topology is not heterogeneous) where relevant, since the “well-mixed” scenario is a special case of lattice structure, where average degree is network size minus one. We focus on a coordination game (sometimes referred to as a “stag-hunt” game), with the view of understanding the evolution of coordination in these topologies. We also focus on a parameter  $\alpha$ , which changes the strength of strategy selection/introduces noise, a model that has also been used in other studies of stochastic strategic interaction [21].

Parameter variation has been utilised to emulate non-linear responses in evolutionary games [22,23]. It has been suggested that global optimisation plays a vital role compared to local optimisation, especially in crises [24]. Interestingly, changing a parameter in an evolutionary game has been compared to selecting a personality type, and this in turn could lead to new equilibrium concepts [25]. We employ these findings in evolutionary game theory and adaptive systems to study the evolution of coordination in networked evolutionary games.

We find that in all topologies that we have considered, it is beneficial to adopt the non-coordination strategy if there is no information diffusion. However, if there is information diffusion, above a particular value of  $\alpha$ , it is more beneficial to be a coordinator, and nodes increasingly adopt this strategy during the evolutionary process. Furthermore, we find that in all topologies, the proportion of coordinators go through a sharp phase transition in terms of the relative payoff for coordination. We also find that the provincial hubs are the drivers of the evolution of coordination, adopting the coordination strategy the quickest. Comparing topologies, we find that the small-world networks show the sharpest phase-transition

in terms of payoffs, and are most robust to outdated information regarding payoffs, while scale-free networks are most sensitive to noise as well as time lags in information propagation.

The rest of this paper is organised as follows. In Section 2, we discuss the justification of choosing particular classes of networks for simulating the coordination game. In Section 3, we explain our simulation set-up. In Section 4, we present our results. In Section 5, we discuss our results and provide a summary.

## 2 Studying games on networks

To analyse the evolution of coordination on realistic network topologies, we utilised the well known “stag-hunt” game, which is an example of a coordination game [23,26,27]. Since it is a two-player game, it can be simulated pair-wise on neighbours sharing a link on a social network model. In this game, if two players coordinate, they both get the highest reward  $S$  (half a Stag), and if one-player does not coordinate while the other does, the non coordinator gets higher reward  $R$  (Rabbit), while the coordinator gets a lower reward  $T$  (Nothing). If both players do not coordinate, they both get reward  $R$  (Rabbit). The game is modelled such that  $S > R > T$ . The analogy is that half a stag would typically have more meat than a rabbit, and hunting a rabbit is better than going home empty handed.

In the classical “stag-hunt” game, coordination is needed only to hunt a stag, whereas rabbit can be hunted independently by each person/agent. As such, a “coordinator” (someone who uses a “coordinating” strategy) is somebody who intends to hunt a stag, and a non-coordinator (sometimes called a “defector”, though this term is more appropriate to the Prisoners Dilemma game) is someone who intends to hunt a rabbit, in a particular round. A typical payoff matrix for the classical stag-hunt game is shown in Figure 1. This game can be applied in strategic decision making situations where two players can gain higher payoff by coordinating rather than by defecting. The pure strategy Nash equilibria of this game occur when both players coordinate and both players defect. We did not consider the mixed strategy scenario in this study.

**Scale-free networks:** it has been recently shown that many real-world networks are scale-free networks, including technical, biological and social networks [16,17,28–34]. Particularly, many social networks are scale-free and heterogeneous, because there are always people who are more “famous” and well-connected, while there are many who are relatively isolated. Scale-free networks display power-law degree distributions, described by  $p_k = Ak^{-\gamma}U(k/k_{\max})$  where  $U$  is a step function specifying a cut off at  $k = k_{\max}$ . There are a number of growth models which generate scale-free networks, and prominent among them is the Barabási-Albert model [15] utilising preferential attachment. Due to the prevalence of scale-free features in many online and offline social networks, scale-free networks are good models to study games on

		Hunter 2	
		Stag	Rabbit
Hunter 1	Stag	S	T
	Rabbit	R	R

**Fig. 1.** The payoff matrix of a typical stag-hunt game. The highest payoff is obtained when both players coordinate. Typically  $S > R > T$ , since it is assumed that half a stag has more meat than a rabbit, and hunting a rabbit is better than not hunting anything. In this paper, without loss of generality, we fix  $S = \beta$ ,  $R = 1$ , and  $T = 0$ , where  $\beta > 1.0$ .

social systems, and often used for this purpose in recent literature [35].

**Small-world networks:** an equally justifiable yet less used model is the small-world network model. Small-world networks have low characteristic path lengths (compared to network diameter) and high clustering [36–38]. The small-world effect on social systems was first and famously demonstrated by Milgram with a network of acquaintances [39] in United States, where he showed that the average number of hops required before a letter addressed to a random addressee within the country reached them was only six: thus the “six-degrees of separation” [40]. It has since been shown that a range of real-world networks, including social networks, biological networks such as gene regulatory networks, metabolic networks, protein-protein interaction networks, and signalling networks, as well as Internet show the small-world property [17,41,42]. Of course, many small-world networks can be scale-free to a certain degree, and vice-versa, but the scale-free and small-world characteristics need not (and often, do not) overlap.

**Hierarchical-modular networks:** another category of networks coming into prominence recently is modular and hierarchical-modular networks. It has been recently observed that the hierarchical-modular structure of brain networks enhances the brain’s robustness [43]. Similarly, many designed and evolved engineered systems are highly modular [44,45]. More importantly, hierarchical-modular structure has been observed in human/social networks as well. For example, Ahn et al. [46] studied hierarchical organisation in several social networks. It is also evident that networks of people in the military/defense domain naturally exhibit hierarchical structure [47]. In the end, hierarchy is inherent in the social structure of human beings, coupled with modularity; therefore it makes sense to study how coordination games can be played in hierarchical-modular networks. Therefore we chose this topology as the third topology of interest.

**Erdős-Rényi random networks:** finally, we also use the Erdős-Rényi random topology [17]. Even though such random networks were once used extensively to model distributed systems, researchers have since realised that most real-world networks do not display degree distributions similar to random networks [15]. Yet, random networks are often used as null models to compare against other network models, and we use them for the same purpose in this work.

**Well-mixed networks:** traditionally, game theory experiments were simulated on “well-mixed” populations [35], where every agent was assumed to be connected to every other agent, before the importance of topology was realised, spawning the research area of networked game theory. We therefore also test some of our results on well-mixed populations for comparison. A network which simulates a well-mixed population is a regular lattice, with average degree of  $N - 1$ , where  $N$  is the number of nodes. We found however, that this average degree has no bearing on the results, and any regular lattice yields qualitatively similar results in the experiments described in the paper. For simplicity, therefore, we present the results obtained from a regular lattice of average degrees four, eight or twelve, so that it matches with the average degrees of other topologies.

Therefore, we use separate scale-free, small-world, hierarchical-modular, random and lattice topologies to study evolution of coordination in social systems.

### 3 Simulation design

In order to simulate the coordination game played on a population of nodes, we used an ensemble of scale-free networks, small-world networks, hierarchical-modular networks, E-R random networks, and lattices. The scale-free networks were generated using a version of preferential attachment [48], varying the average degree of the networks. The small-world networks were generated using the algorithm proposed in Watts and Strogatz [36] using a rewiring probability of  $p = 0.5$  (unless otherwise specified), again varying the average degree of the networks. To produce hierarchical-modular networks, we follow the method described by Sarkar and Dong [49]. This methodology involves ‘rewiring’ each edge in a perfectly modular network to take away intra-community edges in each module with a rewiring probability  $p$ . By varying  $p$ , we obtain networks that have varying levels of hierarchy. The Erdős-Rényi random networks were generated simply by randomly choosing  $M$  pairs among  $N$  nodes and connecting them. Generating lattices is trivial. We typically used network size  $N = 10^3$  and averaged over one hundred networks for each parameter configuration. In evolution scenarios, we typically considered  $T_e = 1500$  timesteps to be sufficient for the network to achieve steady state. This number was chosen based on the preliminary results.

The payoff matrix of the game was constructed in such a manner that the reward for both parties coordinating  $S$  would be a variable  $\beta$ , such that  $\beta > 1$ . When one node is

coordinating and the other is not coordinating, the coordinator would not get any return ( $T = 0$ ) while the non-coordinator would get a return of unity ( $R = 1$ ). Thus, we could manipulate the game environment by varying the single parameter  $\beta$ . Each pair of nodes connected by a link would engage in a single round of coordination game, after which the collective returns for each node  $p_i$  would be stored and used to adjust the accumulated payoff,  $P_i$ .

At the beginning, players were randomly assigned as coordinators or non-coordinators. After each iteration, the players would adopt the role of the neighbours based on a certain probability. This probability would be affected by the current accumulated payoff of each node. In the case of complete information diffusion, suppose that two nodes  $x$  and  $y$  are connected and their current accumulated payoff values are  $P_x$  and  $P_y$ . These are the payoffs that are accumulated within each node after a certain number of timesteps. If  $x$  and  $y$  are different in their respective roles (that is, one is a coordinator and the other is a non-coordinator), the probability  $p$  that  $x$  would adopt the role of  $y$  is given by:

$$p = \max \left\{ 0, \frac{(P_y - P_x)}{k_{\max}(R - T)\beta} \right\} \quad (1)$$

where  $k_{\max}$  is the larger of the degree of  $x$ ,  $k_x$  and the degree of  $y$ ,  $k_y$ . This is a model commonly used in recent literature [35] to simulate evolutionary adaptation in a game scenario<sup>1</sup>.

We modified this model to quantify information diffusion, by introducing a parameter  $\alpha$  which signifies the level of information a node can gather about its neighbours. Therefore, a node may change strategies either (i) randomly (ii) based on information of its neighbours' payoffs. Therefore, we model the adaptation probability as:

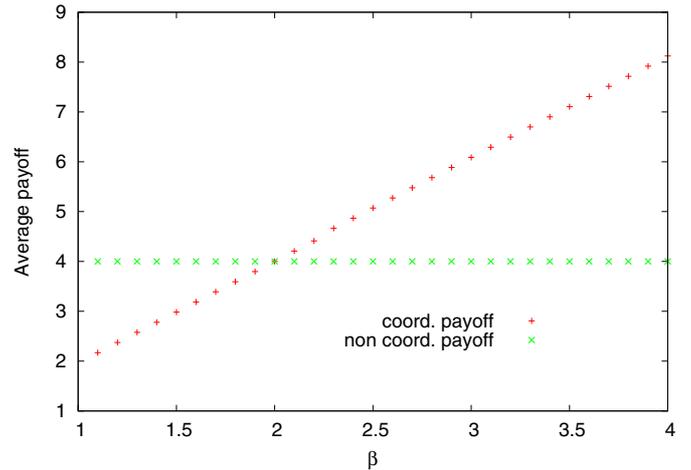
$$p = (1 - \alpha)\rho + \alpha \max \left\{ 0, \frac{(P_y - P_x)}{k_{\max}(R - T)\beta} \right\} \quad (2)$$

where  $\rho$  is a uniformly distributed random number between zero and one, and represents white noise. Therefore, the higher the  $\alpha$ , the more the ability of the system to distinguish real cumulative payoff information from noise.

Later in the paper, we also analyse the influence of time-lags on the information diffusion and the emergence of coordination. Therefore, we introduce a time-lag  $\lambda$  and the pay-offs considered are those payoffs which each node had accumulated  $\lambda$  timesteps before the current time,  $P_x^\lambda$  and  $P_y^\lambda$ . As such, the diffusion equation becomes

$$p = (1 - \alpha)\rho + \alpha \max \left\{ 0, \frac{(P_y^\lambda - P_x^\lambda)}{k_{\max}(R - T)\beta} \right\}. \quad (3)$$

<sup>1</sup> However, note well that [35] uses different symbolism, with a pay-off ordering where  $T > R > S$ , and their application is to a Prisoners Dilemma game, thus the update rule we propose may seem slightly different at first glance. However, simple analysis will reveal that the rules are essentially very similar.



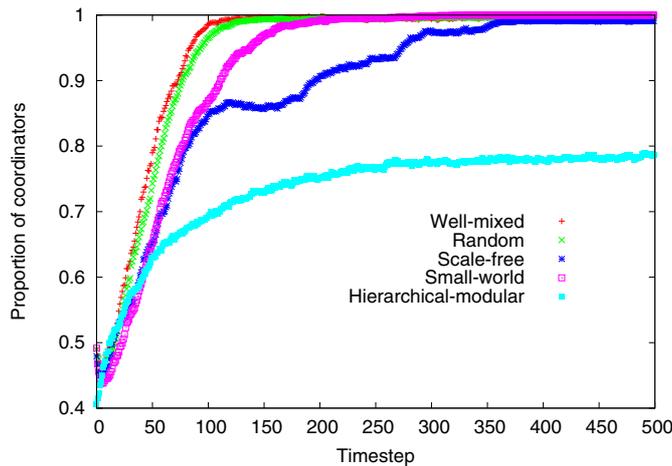
**Fig. 2.** The average score of coordinators and non-coordinators in 100 scale-free networks after a single round of coordination game. Note that results for  $T_e = 1500$  timesteps were identical (results do not vary with time). The nodes do not have a priori knowledge about the strategy used by their neighbours. It can be seen that for coordination rewards  $\beta \leq 2.0$ , the average reward for non-coordinators is higher. Similar results were obtained for small-world, hierarchical-modular, random network, and regular lattice classes.

## 4 Simulation results

### 4.1 Pre-evolutionary balance

First of all, we set out to understand the balance between payoffs of the coordinators and non-coordinators on average, when nodes do not adapt. Therefore we simulated single rounds of coordination game on scale-free, small-world, hierarchical-modular, random and lattice networks, without any evolution. In the case of scale-free networks, one-hundred networks (with a size  $N = 1000$  and an average degree  $\bar{k} = 4$  in all cases) were used, and we calculated the average payoffs for coordinators and non-coordinators for a range of  $\beta$  values. The results are shown in Figure 2. When we undertook the same process for  $T_e = 1500$  rounds, the results were identical: understandably, since, in the absence of evolution, number of rounds would not make any difference, and the results were averaged across networks anyway.

From the results, we can see that, predictably, the relative payoff for coordinators steadily increases with  $\beta$ . Only when  $\beta > 2$ , is the average payoff for coordinators higher than that of non-coordinators. Therefore, the *total payoff* for a pair of coordinators must be four times higher than the individual return of a non-coordinator (the stag must have four times more meat than the rabbit) for people in a scale-free network to decide to adopt the coordination strategy, in the absence of information about strategies adopted by others and their payoffs. If  $\beta = 1.5$  for example, (the stag has three times more meat than the rabbit), it may intuitively seem better to coordinate, since half a stag still has more meat than a rabbit, yet due to the chance of the other node (person) not-coordinating, this is



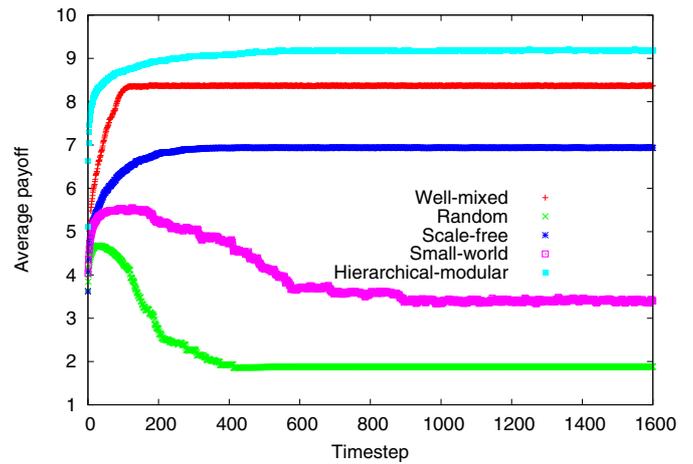
**Fig. 3.** Proportion of coordinators against the timestep for the five different types of networks considered. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps are considered (though only up to 500 are shown in figure, since the trend is clear after this point).  $\beta = 1.8, \bar{k} = 4$ .

not the case. This is an important observation. We found very similar results for the small-world, E-R random and hierarchical-modular networks (not shown), therefore it is also obvious that this result does not depend on the network topology. We also analysed regular lattices of various average degrees (starting from four up to  $N - 1$ , where  $N$  is network size), and found that this result ( $\beta > 2$  for coordinators to have higher relative payoff) holds also for well-mixed populations: not surprising, since this simply is a consequence of the relative proportion of coordinators and non-coordinators in the population (50% each). As such, it is clear that in the absence of evolutionary adaptation, topology does not determine the relative payoffs of coordinators and non-coordinators (when the network concerned is sufficiently large to negate finite-scale effects).

#### 4.2 Evolution of coordination

Now, we simulate evolution of coordination. First, we assume complete information diffusion and use equation (1) to simulate the evolution of strategies in all five classes of networks. We used three different average degree values ( $\bar{k} = 4, 8, 12$ , respectively), and utilised the average over 100 networks in each case. We simulated evolution for  $T_e = 1500$  timesteps for each network, and measured the proportion of coordinators during and after the evolution.

Figure 3 shows the evolution of proportion of coordinators, for networks of  $\bar{k} = 4$ , and  $\beta = 1.8$  was used. It could be seen that in all classes of networks, coordinators begin to dominate after a certain stage. We have seen earlier that in pre-evolutionary balance, for a  $\beta$  less than two, it is advantageous to be a non-coordinator. However, it appears that in evolved systems where players are aware of the payoffs of their neighbours, it is advantageous to be a coordinator after a certain time frame. This is confirmed



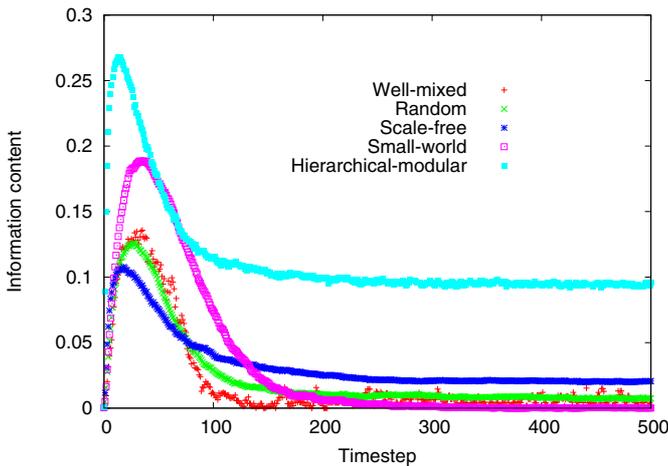
**Fig. 4.** Average payoff of coordinators against the timestep for the five different types of networks considered. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps are considered.  $\beta = 1.8, \bar{k} = 4$ .

by Figure 4, which shows the average payoff of coordinators against timestep, for  $\beta = 1.8$ , and for all five classes of networks. In all cases, this average payoff initially increases with time, though it decreases in some cases once coordinators become a majority. Obviously, for  $\beta \geq 2.0$ , it is advantageous from the beginning to be a coordinator, and we found that this is not changed by the evolution of the system.

Interestingly, we also find from Figure 4 that while for scale-free and hierarchical-modular networks (as well as the well-mixed population), the average payoff of coordinators increases and stabilises with evolution, this is not the case with small-world and E-R random networks. With these networks, the average payoff for coordinators increases, then decreases and stabilises. Considered with Figure 3, it is clear that in these two classes of networks, the non-coordinators become extinct after a certain number of timesteps. Thus, in these networks, the lower average payoffs are derived at larger timesteps when all players within the network are coordinating.

We note however, that there is not much difference in the time taken for the coordinators to dominate, between the classes of networks that we have studied. However, in the case of hierarchical-modular networks, non-coordinators are able to survive and hold a proportion of the network. In all other topologies, non-coordinators are “wiped out” by evolution.

To complement this analysis, we also looked at the “information content” of the node states in each class of networks, and how they evolved as the simulation progressed over time. Shannon information is a generic measure of “information content” in a system. In reference [50] an information content measure  $I(q)$  was defined based on Shannon information for complex networks and on the remaining degree distribution of the network  $q_k$ , and [51] extended this definition so that node states are considered. The mutual information measure defined in



**Fig. 5.** Information content in the network against the timestep for the five types of networks considered. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 500$  timesteps are considered.  $\beta = 1.8$ ,  $\bar{k} = 4$ .

reference [51] is

$$I(q^t) = \sum_y \sum_z e_{y,z}^t \log \frac{e_{y,z}^t}{q_y^t q_z^t} \quad (4)$$

where  $e_{y,z}^t$  is the proportion of links connecting, at time  $t$ , the nodes with states  $y, z$ , respectively;  $q_y^t$  is the proportion of links, at time  $t$ , with a node (at one end) in the state  $y$ ; and similarly,  $q_z^t$  is the proportion of links, at time  $t$ , with a node (at one end) in the state  $z$ . We used this measure to analyse how the mutual information in terms of node states (coordinator/non-coordinator) changes during simulation, and our results are presented in Figure 5. Interestingly, we see that the information content increases rapidly during the initial stages of evolution, and peaks at a point much earlier than when coordinators saturate the networks in the respective class. It appears that the timestep when the mutual information peaks is the timestep when the coordinators “break-through”, when they attain a critical number after which their eventual complete domination becomes inevitable. However, when coordinators completely dominate, the mutual information content is close to zero. This is not surprising since there is no “information” left to be gained regarding the node status with respect to topology. Subsequently, we note that the steady state information content for hierarchical networks is greater than the other classes, since we saw earlier that the domination of coordinators is not complete in this class and some non-coordinators manage not to convert.

Now we consider how the payoff parameter,  $\beta$ , influences the domination of coordinators. For this, let us consider each class of networks separately. In the case of scale-free networks, we used three different average degree values ( $\bar{k} = 4, 8, 12$ , respectively), and again simulated evolution for  $T_e = 1500$  timesteps for each network, and measured the proportion of coordinators *after*

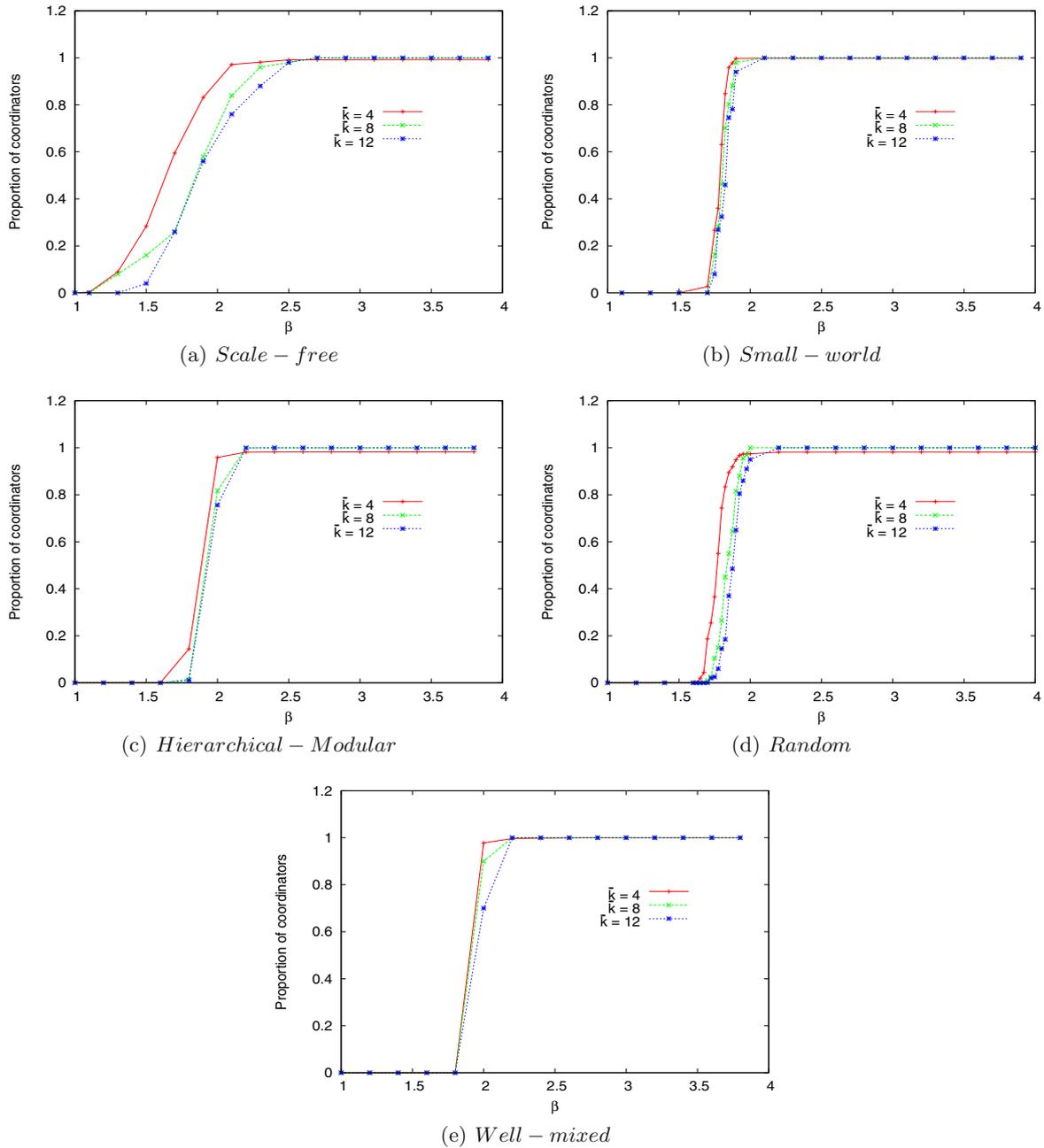
the evolution in each case. Our results against various  $\beta$  values are shown in Figure 6a.

We may see from this figure that, even though non-coordination is initially the better strategy, coordination emerges as the better strategy and adopted by a majority of nodes after a period of time, for a range of  $\beta$  values less than 2.0. Moreover, there is phase transition in terms of  $\beta$ , which occurs when  $\beta \leq 2.0$  in most cases. For example, when  $\bar{k} = 4.0$ , it appears that coordination is a better strategy and adopted by more nodes eventually, if  $\beta \geq 1.6$ . Therefore, we can come to the important conclusion that there is a range of  $\beta$  values (e.g.  $2.0 \geq \beta \geq 1.6$  for networks with average degree 4.0), for which it is beneficial to adopt the non-coordination strategy if there is no information diffusion, however coordination is the evolutionarily winning strategy (and the evolutionarily stable strategy, ESS) if there is information diffusion about the cumulative payoff of the neighbours. However, if the relative payoff of coordination is sufficiently low (but higher than payoffs for non-coordination, e.g.  $1.6 \geq \beta \geq 1.0$  for networks with average degree 4.0), it is evolutionarily better strategy to adopt non-coordination. We can also observe that for higher network density, it takes higher rewards of coordination for coordinators to become dominant.

Now we undertake a similar analysis in the case of small-world networks, to ascertain the influence of payoff parameter  $\beta$  on the evolution of coordination. We generated small-world networks with size  $N = 1000$  nodes and varying average degrees ( $\bar{k} = 4, 8, 12$ ). We again simulated 100 networks in each case, for number of time steps  $T_e = 1500$ . We find similar results to scale-free networks in small-world networks, as shown in Figure 6b. That is, coordination emerges as the better strategy through evolution for higher values of  $\beta$ . We find one important difference though. The phase transitions we observe in terms of  $\beta$  are much sharper compared to scale-free networks. Therefore, a slight increase in the actual amount of payoff can very quickly change the evolutionary dynamics of coordination in small-world networks. The phase transition, for this particular set of parameters, seems to occur around  $1.9 \geq \beta \geq 1.7$ .

We may also observe that, comparing the results between various  $\bar{k}$  values, for both scale-free and small-world networks, the phase transition begins “later” for higher average degrees, and coordinators begin to dominate only for higher values of  $\beta$ . For example, in the case of scale-free networks, when the average degree is 4.0,  $\beta$  is approximately 1.6 when the coordinators begin to dominate, while for average degree of 8.0, coordinators begin to dominate for  $\beta = 1.8$  approximately. Even though the difference is small, we can come to the important conclusion that when more links (relationships) are added to an existing network, increasing link density and average degree, it takes higher relative rewards for coordination, for it to become the evolutionary dominant strategy. In a more densely connected society, the “stag” has to have relatively more “meat” for hunters to adopt coordination.

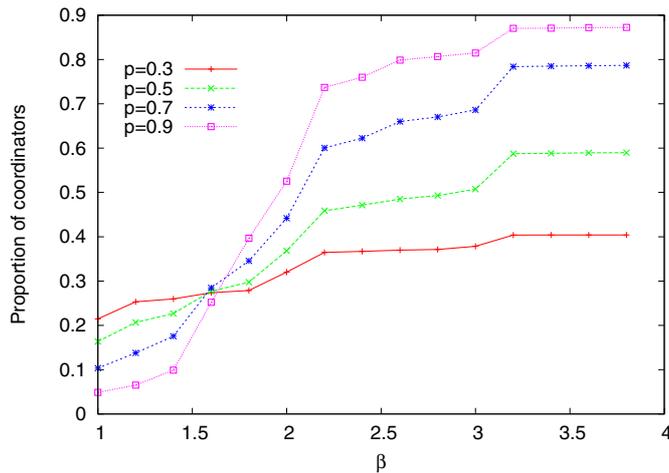
We now turn to hierarchical-modular networks. As Figure 6c shows, here also there is a sharp phase



**Fig. 6.** Proportion of coordinators against the game parameter  $\beta$  for five classes of networks with different average degrees, after evolution. It can be seen that for each class of network there is a value  $\beta$  above which coordination dominates non-coordination. For low  $\beta$  values, non-coordinators dominate in all networks. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps were used for evolution.

transition. Does hierarchy in social systems favour the evolution of coordination? To answer this, we undertake further analysis using the hierarchical-modular networks generated by the method described in Sarkar and Dong [49]. Namely, we generated ensembles of 100 networks each for various values of wiring parameter  $p$ , from 0.3 to 0.9. Sarkar and Dong explain that the higher this parameter, the higher the hierarchical nature of the networks. For each set of networks, we simulated evolution

of coordination as described before and measured the proportion of coordinators at the end of simulation (after  $T_e = 1500$  timesteps) against the payoff parameter,  $\beta$ . Figure 7 shows some of our results. We observe from the figure that hierarchy indeed aids the dominance of coordinators in a certain way. While coordinators do not dominate below a certain payoff parameter ( $\beta = 1.6$ ), and the value of this cut-off is not influenced by the parameter  $p$ , the evolutionary behaviour for values higher than this  $\beta$

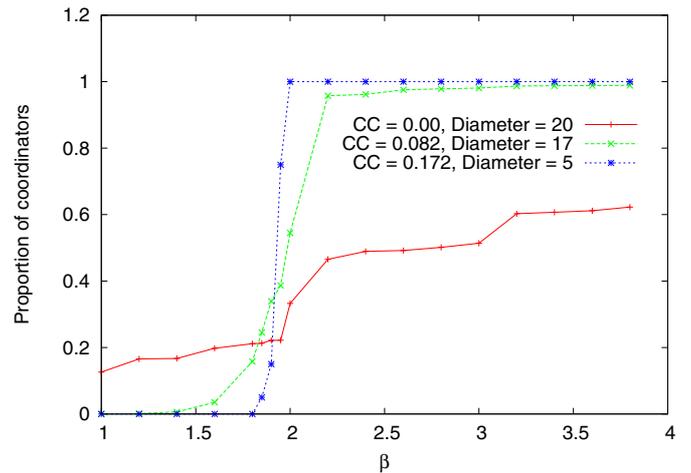


**Fig. 7.** Proportion of coordinators against the game parameter  $\beta$  for hierarchical-modular networks with different rewiring  $p$  probabilities, after evolution. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps were used for evolution.

is influenced by the amount of hierarchy represented by  $p$ . For lower values of  $p$ , the coordinators do not dominate at all for any  $\beta$ . However, for higher values of  $p$ , the coordinators dominate and a phase transition is vaguely observable around  $\beta = 1.6$ . Therefore, we may conclude that other topological features being similar, the presence of hierarchy encourages the emergence of coordinators (given sufficient relative payoff) in a social system.

Similarly, we may ask if “small-worldness” of small-world networks encourages or discourages the evolution of coordination. The “small-world” nature is quantified by (i) relatively high clustering coefficient, (ii) relatively low characteristic path lengths of a network [15,17]. Therefore, we also analysed how these two parameters influenced the evolution of coordination in small-world networks. Some typical results are shown in Figure 8. We could see from the figure that, when clustering coefficient increases and network diameter decreases in a set of similar sized ( $N = 1000, M = 2000$ ) small-world networks, again generated using the Watts- Strogatz algorithm, the phase transition in terms of relative coordinator payoff ( $\beta$ ) becomes more pronounced. Thus, we may conclude that small-world nature again encourages rapid evolution of coordination, when the relative payoffs for coordinators increase.

Backing and intuitive explanations for these observations may be found in other recent studies, which looked at other games on graphs. For example, Masuda and Aihara [52] found that cooperative behaviour in spatial Prisoners Dilemma (similar to coordination in stag-hunting) is optimised when network is small-world. They arrived at this conclusion by comparing a range of graphs from regular lattices to random graphs, and the small-world characteristic was determined by the amount of randomness introduced into the lattice (as explained by Watts and Strogatz [36]). The topology was shown to be most conducive to cooperative behaviour when the randomness

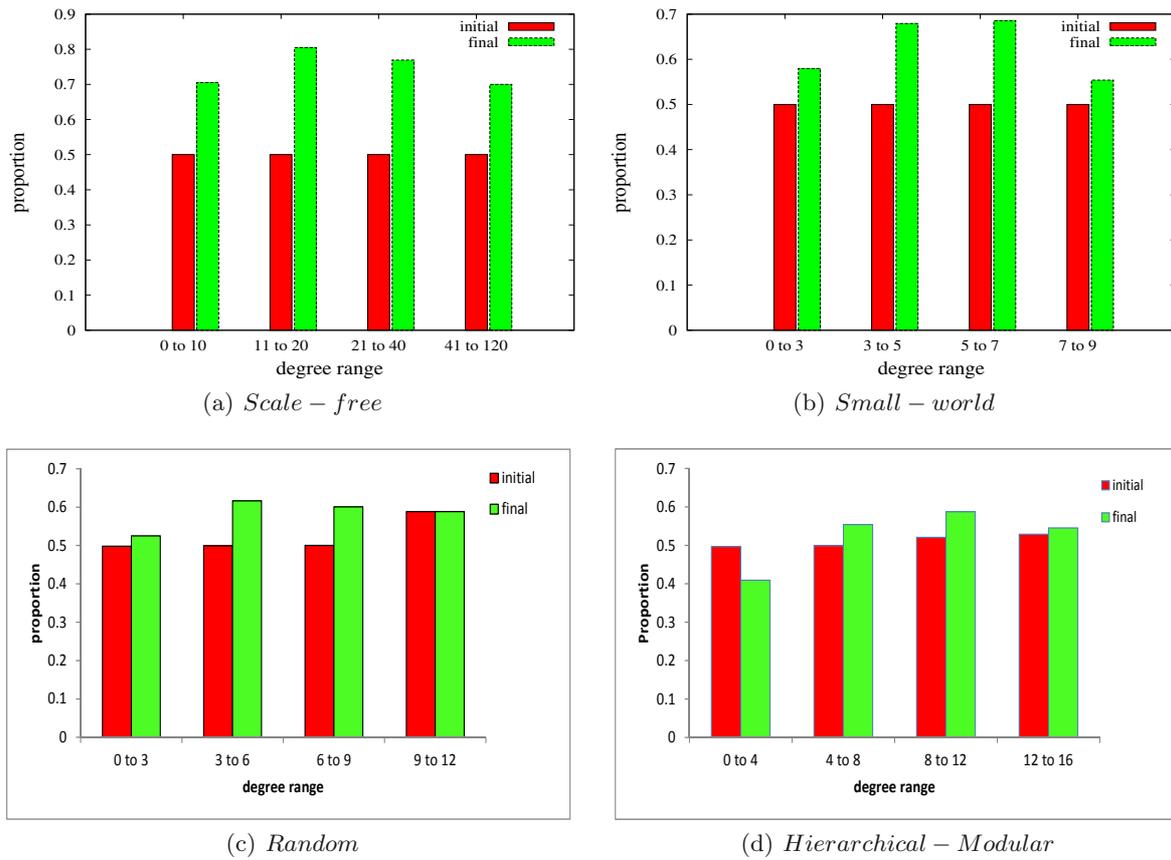


**Fig. 8.** Proportion of coordinators against the game parameter  $\beta$  for small-world networks with different (a) network diameters, (b) clustering coefficients after evolution. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps were used for evolution.

parameter was intermediate in value, which also maximises “small-worldness”. On the other hand, in a hawks and dove game, where coordination does not result in the highest payoff for an individual (the best scenario for an individual is to be the hawk themselves while the other player plays dove, whereas both players playing hawk in fact get a negative payoff), no such optimisation for coordination was observed in small-world networks [53]. The reason for these observations may be that, in games where mutual cooperation/coordination is not detrimental (in stag-hunting it results in best possible payoff, whereas in prisoners dilemma it results in the second best possible payoff for an individual), the high clustering introduced by “small-worldness” helps sustain a group of coordinators/cooperators while the “short-cuts” available in topology help it spread. If the network is lattice like, there are no short-cuts to further parts of the graph, hindering the spread of coordinators, while if it is totally random, the clustering is lost, making it harder for coordinators to sustain each other.

Similarly, in terms of modularity and hierarchy, we may note that in networks which are highly modular, most nodes belong to certain “groups” (modules), while there are other links which maintain the hierarchical structure. Recent work has shown that there is an optimal proportion of inter-group links, for which the spreading of cooperative behaviour for Prisoners Dilemma is optimised [54]. Assuming a similar phenomena occurs in the case of the coordination game, it is possible that the increase in hierarchy as proposed by Sarkar and Dong [49] moves the fraction of inter-module links towards this optimal proportion, thus facilitating the spread of coordination. Detailed examination of this particular relationship is of interest; however, it is beyond the scope of this paper.

Finally, the case with E-R random networks is shown in Figure 6d, and well-mixed populations in Figure 6e. Here, too, we could observe a sharp phase transition,



**Fig. 9.** Evolution of coordinators by degree for four classes of networks (regular lattices are not considered since there is no variation of degree in them). The figure shows the fraction of coordinators for a number of degree intervals, averaged over 100 networks, at the beginning and at an intermediate time  $T_i$  evolution. Since strategies were initially randomly assigned, the proportion of coordinators is about half for each degree interval at the beginning. At the intermediate time however, the proportion of coordinators is much higher overall but highest among the provincial hubs. The network size was  $N = 1000$  nodes in all cases, and  $T_i = 100$  timesteps were used for evolution,  $\bar{k} = 4$  and  $\beta = 2.1$ .

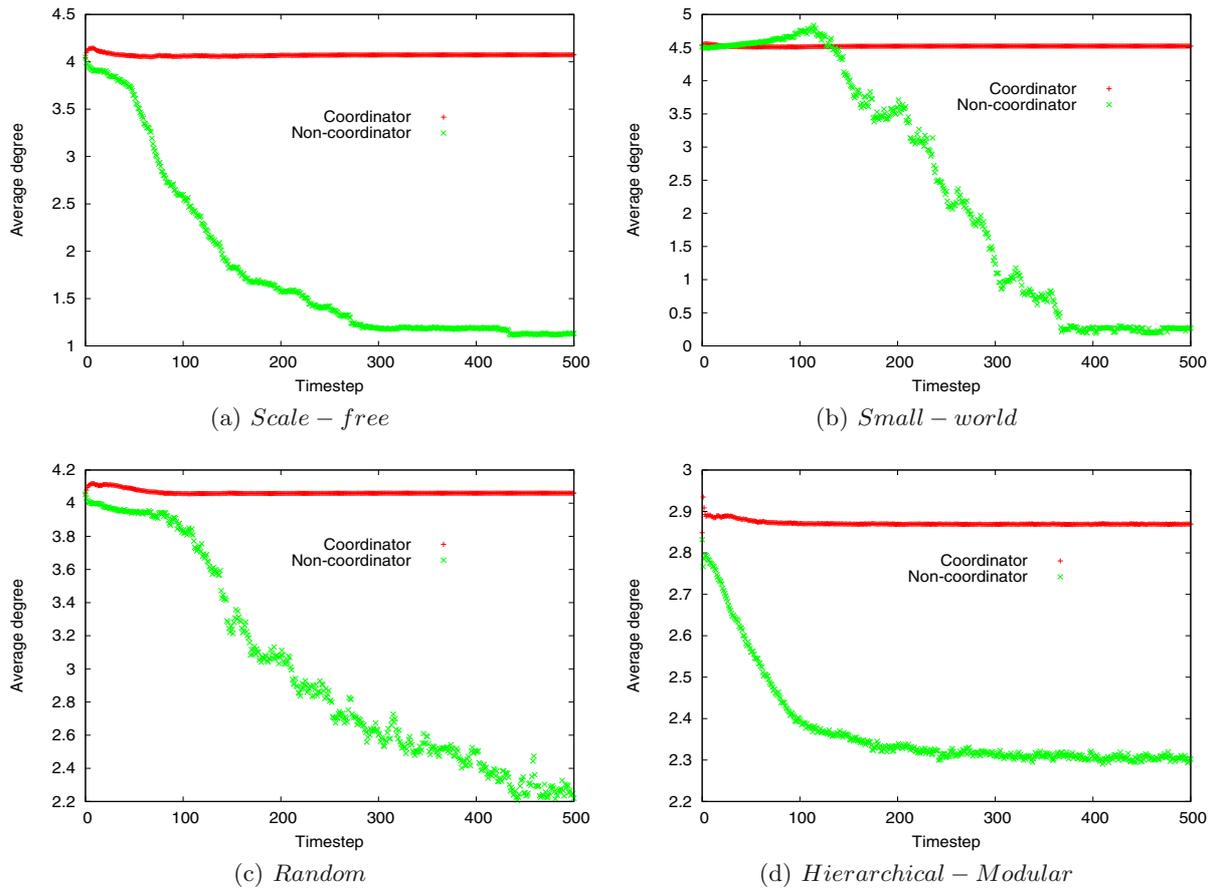
around  $1.9 \geq \beta \geq 1.7$ . The average degree of network does not seem to influence much where this transition occurs. Therefore, we may conclude that while all classes of networks display some degree of phase transition in terms of  $\beta$ , it is sharpest in small-world networks. This is an important result, since it means that in a small-world network, with a scenario of increasing relative payoffs for coordinators, the decision to become coordinator has to be made quite swiftly in order not to lose out. On the other hand, in a society where social links are scale-free, it is possible to decide more “slowly” (given that coordinator payoffs increase at a fixed temporal rate) about becoming a coordinator.

### 4.3 Drivers of coordination and node degree

To gain some understanding of how coordination ends up being the winning strategy, we analysed the degree distribution of coordinating and non-coordinating nodes during the process of evolution. An example is shown in Figure 9 where we consider the average of 100 networks

with  $\bar{k} = 4.0$ , for a  $\beta$  of 2.1. Therefore, according to Figure 6, these are networks on which coordination dominates after  $T_e$  timesteps. We deliberately chose an intermediate timestep,  $T_i = 100$ , with the view of understanding which degree range is first dominated by the coordinators. In case of scale-free networks, we find that it is the provincial hubs, which first start to show higher proportion of coordinators. It appears that main hubs resist the adaptation longer, and once they become predominantly coordinators, the evolution of coordination is almost complete. Some of the peripheral nodes also can remain non-coordinators for a long time. It is the provincial hubs, which are the quickest to adapt.

In the case of small-world networks, we again looked at the degree distribution of the network in terms of strategy at an intermediate time step ( $T_i = 100$ ). The results, averaged over 100 networks of thousand nodes each, are shown in Figure 9b. Note that the small-world networks, by nature, have much smaller hubs compared to scale-free networks. Again we find that it is the provincial hubs, which seem to be first adopting the coordination strategy completely. We obtained similar results for



**Fig. 10.** Average degrees of coordinators/non-coordinators against timestep. Four different network types are considered.

hierarchical-modular and random networks, as shown in Figures 9c and 9d. We ignored regular lattices since node degrees are homogenous in them and the analysis is unnecessary. That is, in all cases, it is the provincial hubs which are first fully “converted” into becoming coordinators.

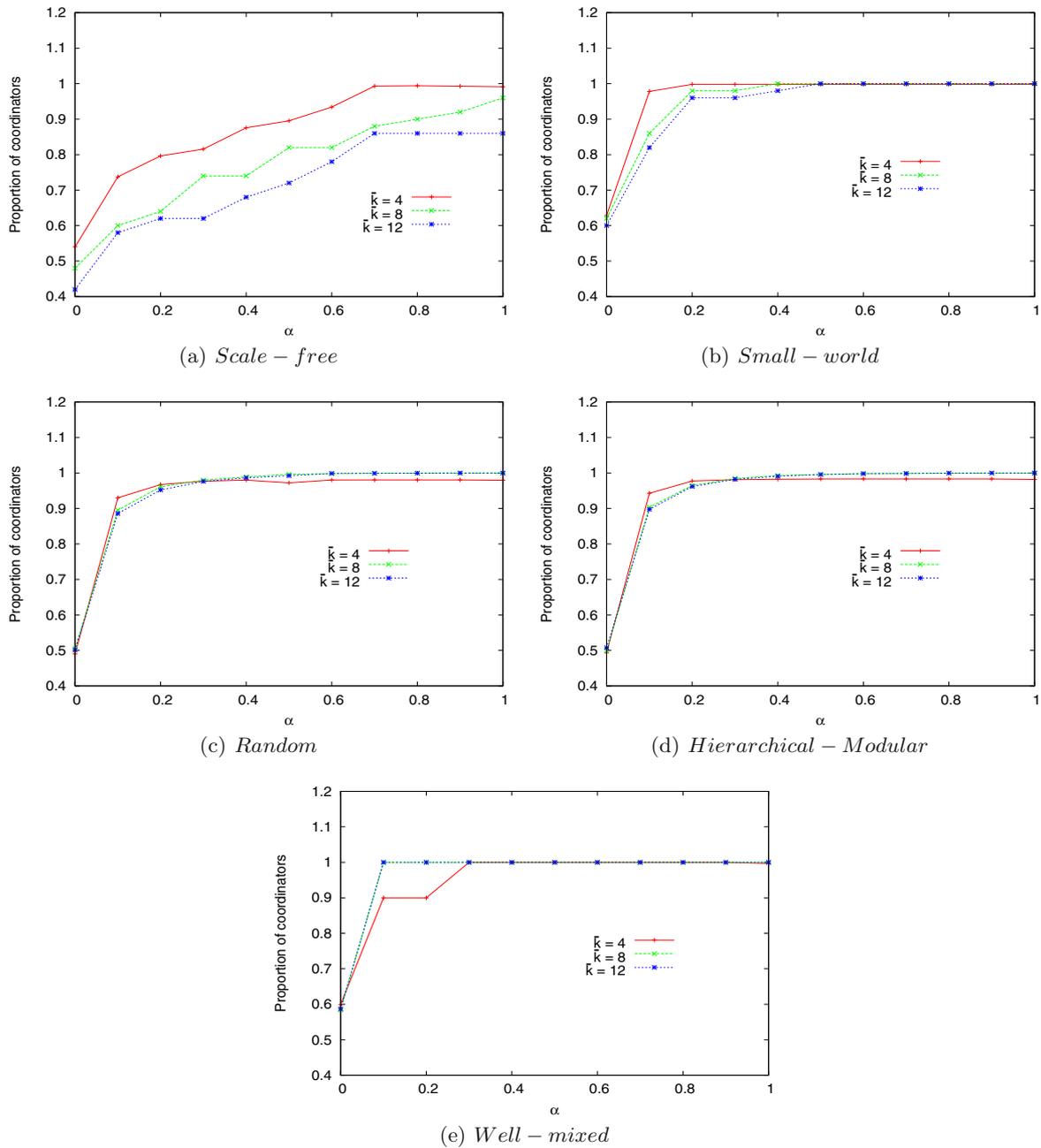
Now let us consider the question of which types of nodes “convert” first to coordination. To do this, we plot the average degree of coordinators and non-coordinators throughout evolution for each class of networks (again averaged over 100 networks of  $N = 1000$  each) in Figure 10. Here we can see that while the average degree for coordinators remain more or less the same, the average degree of non-coordinators decline steadily. This is not inconsistent with provincial hubs first adopting coordination, because it means that once provincial hubs (which have degrees higher than the network average) start becoming coordinators, the average degree of non-coordinators begins to decrease. However, we notice a special feature in terms of small-world networks. In this class, the average degree of non-coordinators briefly becomes higher than the average degree of coordinators, before the average degree of non-coordinators decreases rapidly. This can only mean that in small-world networks, non-coordinators find refuge in main hubs (while provincial hubs are “invaded” by coordinators) and resist the spread of coordinators for a while, before giving up and retreating to a few peripheral nodes.

It is intriguing that players who play a large number of games tend to adapt slower in small-world networks than in other topologies.

#### 4.4 Influence of information diffusion

It is often unrealistic to expect that members of a community would know, or can correctly predict, the strategies adopted by their neighbours or the payoffs they are receiving. Indeed, as we saw earlier, a non-coordination strategy is transiently competitive and pays better dividends in the short term, so many members of the community may believe that it is not in their interests to share correct information about their strategies or their payoffs. If nodes only have partial information about payoffs of their neighbours, which is the evolutionarily competitive strategy in each of the network classes mentioned before? This is the primary question that we are addressing in this section.

We therefore use equation (2) to introduce stochasticity in the adaptation. Nodes have “noise” in the information which they have about the payoffs of their neighbours, and therefore have a level of randomness in changing decisions. The lower the parameter  $\alpha$ , the higher the randomness. If  $\alpha = 0$ , no correct information diffuses about



**Fig. 11.** Influence of the information diffusion parameter,  $\alpha$ , on the proportion of coordinators during evolution. The figure shows that the more information available about neighbour strategies, the higher the likelihood of coordinators dominating. If the levels of information available is relatively very low, non coordinators dominate even after evolution. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps were used for evolution,  $\bar{k} = 4, 8, 12$  and  $\beta = 2.3$ .

neighbours’ payoffs and all decisions to change strategy are made randomly.

We again simulated evolution under these conditions on the five classes of networks that we studied. We start with scale-free networks of size  $N = 1000$ , and the simulation was done for  $T_e = 1500$  timesteps, for various  $\alpha$  values. A typical set of results (for  $\beta = 2.3$ ) is shown in Figure 11a. We saw earlier that, for this  $\beta$ , under complete information diffusion the coordinators will dominate

after some time. Therefore, we compare the proportion of coordinators for a range of  $\alpha$  values.

We predictably find that the proportion of coordinators increases with  $\alpha$ . We also find that there is a very small, but non-zero *alpha* value below which the non-coordinators dominate. Therefore, we can surmise that if the levels of information diffusion is sufficiently low then it is an evolutionarily winning strategy to be a non-coordinator. We undertook similar experiments with other

$\beta$  values. While the starting points of plots vary with these  $\beta$  values (for smaller  $\beta$ , non-coordinators dominate for larger ranges of  $\alpha$ ), the results were qualitatively similar.

In the case of the small-world networks, we also found that the proportion of coordinators increased with information diffusion proportion  $\alpha$ . However small-world networks were able to adopt coordination with smaller amounts of neighbour payoff information (smaller  $\alpha$ ), and the transition was sharper. Figure 11b shows some of our results, again for  $\beta = 2.3$ , for hundred networks of size  $N = 1000$  each after  $T_e = 1500$  timesteps. We can find that for  $\alpha \geq 0.2$ , coordinators completely dominate. Therefore small-world networks seem even more resilient to noise in neighbour payoff information than scale-free networks.

Our experiments with E-R random networks, hierarchical-modular networks, and well-mixed lattices produce similar results, as shown in Figures 11c–11e.

#### 4.5 Influence of time-lag in information diffusion

In many real-world scenarios, information about the strategies adopted by other players, or the payoff received by them, is not immediately available, since players tend to think that it is advantageous to hold such personal information secret. However, with the passage of time, the strategies adopted by players in the past and the payoffs received by them may become available, and this outdated information may partially help players to decide their current strategies. Even if players do not deliberately withhold information, it takes time for information to be transmitted and received. In other words, information about payoffs received by other players tend to have a time-lag. In this section, we concentrate on analysing how such a time lag influences the evolutionary patterns.

We therefore modified the equation (2) such that there is a lag in the payoffs known, which gave us equation (3). For ease of reference, we repeat it here:

$$p = (1 - \alpha)\rho + \alpha \max \left\{ 0, \frac{(P_y^\lambda - P_x^\lambda)}{k_{\max}(R - T)\beta} \right\} \quad (5)$$

where  $P_y^\lambda$ ,  $P_x^\lambda$  are the cumulative payoff values of players  $x$  and  $y$ ,  $\lambda$  timesteps before the present time. Even though a particular player may know their current cumulative payoff, it might make more sense for cumulative payoffs from the same time-step to be compared, since we use the lag  $\lambda$  for both nodes. For  $\alpha = 1.0$ , our results for scale-free networks, small-world networks, hierarchical-modular networks, random networks and lattices are shown in Figures 12a, 12b, 12c, 12d, 12e, respectively. All these figures are for  $\beta = 2.1$  and  $\bar{k} = 4$ . The results were obtained after 1500 timesteps.

We have seen earlier that, if there is no time lag, then coordinators dominate the network after a certain number of timesteps for this particular  $\beta$ . However, we could see from the figures that if there is time lag, the dominance of coordinators is less pronounced. Beyond a certain amount of time lag, non coordinators become the dominant players. Importantly, the effect of time lag depends on the

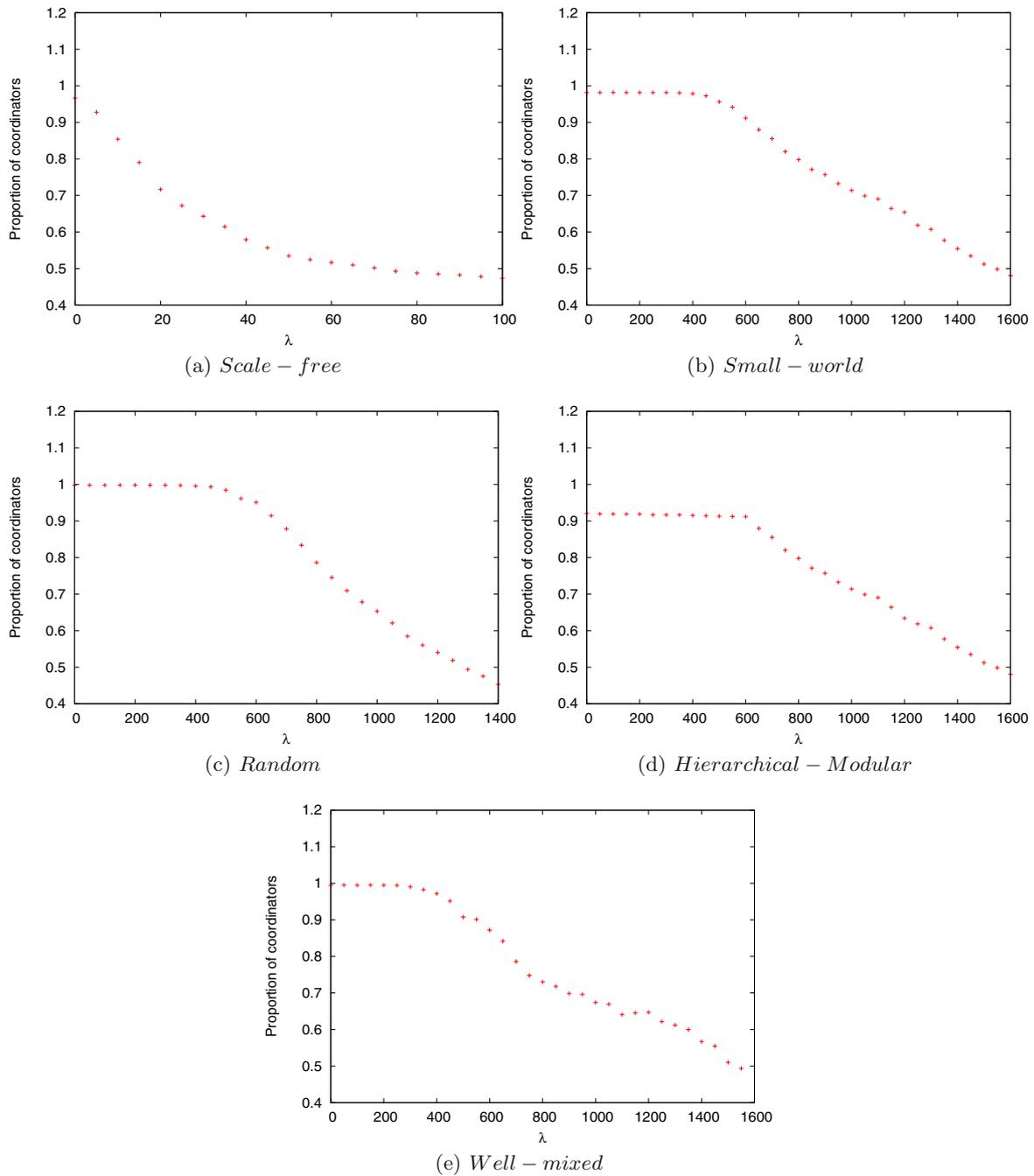
topology of the network. The evolutionary dynamics of scale-free networks change quickly, so that if the time lag is higher than 100 timesteps, then non-coordinators become the dominant players even after evolution. Whereas in the case of E-R random networks, hierarchical-modular networks and small-world networks, and indeed well-mixed populations, the time lag has to be much larger before the non-coordinators dominate evolved networks. The scale-free networks are the least resilient in coping with time lags in payoff information.

Since most real world networks are scale-free [15,17,55–57], it is a very important observation that scale-free networks are more sensitive to time-lags in payoff information than almost any other conceivable topology. The inherent heterogeneity of the degree distribution of scale-free networks could be a reason for this. This question could be further analysed by varying the amount of heterogeneity of synthesized scale-free networks by, for example, changing the scale-free exponent  $\gamma$ , and analysing the value of threshold  $\lambda$  after which each network fails to evolve coordination. The  $\lambda_{\text{threshold}}$  vs.  $\gamma$  plot would then give us an indication about the relationship between time-lag sensitivity and scale-free nature. Such a detailed analysis, however, is beyond the scope of this preliminary study and subject to future research.

## 5 Conclusions and future work

In this paper, we analysed evolution of coordination in social systems by simulating a coordination (“stag-hunt”) game on an ensemble of complex networks. We undertook a comparative study of topologies that are commonly found in social systems, by focussing on (i) scale-free networks, (ii) small-world networks, (iii) hierarchical-modular networks, (iv) Erdős-Rényi random networks. We also considered well-mixed populations or lattices which approximate them, where relevant. In all classes, we saw that if nodes are unaware of the payoffs of their neighbours and cannot adapt, the relative payoff for coordination has to be quite high, for the average payoff of coordinators to be higher than the average payoff of non-coordinators. However, when nodes are aware of the payoffs of their neighbours are receiving and can evolutionarily adapt, coordination quickly emerges as the winning strategy, even for relatively lower levels of coordination payoff.

We came up with a number of general and topology-specific findings which can be summarised as follows. **General findings:** (i) when there is no evolution, the relative coordinator payoff,  $\beta$ , has to be above two for coordinating nodes to have higher average payoff than non-coordinators. This result is independent of the topology of the system. However, after payoff information based evolution and adaptation, there emerges a range of  $\beta$  less than two for which coordinators are still a majority. (ii) In most topologies, the proportion of coordinators after sufficient evolution and adaptation goes through a phase transition when the relative coordinator payoff,  $\beta$ , is increased. (iii) It is the peripheral hubs, which first completely adopt coordination and drive the evolution of



**Fig. 12.** Proportion of coordinators against the time delay parameter  $\lambda$  for five classes of networks after evolution. The network size was  $N = 1000$  nodes in all cases, and  $T_e = 1500$  timesteps were used for evolution.  $\beta = 2.1, \bar{k} = 4$ .

coordination. (iv) Noise and time-lags in payoff information adversely affect the evolution of coordination, though the level of this effect depends on topology.

**Topology-specific findings:** (i) the evolution of coordination is most pronounced, and the phase transition in terms of relative coordinator payoff,  $\beta$ , is sharpest, in small-world networks. On the other hand, the emergence of coordination after evolution is least rapid in scale-free networks. (ii) Scale-free networks are most sensitive to noise in payoff information, and the evolution of coordinators is most affected by such noise in them.

Small-world networks are not so sensitive. (iii) Similarly, scale-free networks are most sensitive to time-lags in information about payoff. (iv) After the evolution of coordination, the average payoff of coordinators is higher than the initial stage in scale-free and hierarchical networks. On the other hand, it is lower than the payoff at initial stage for small-world and E-R random networks. Note however that the proportion of coordinators is higher than initial stage in all classes. (v) The “amount” of hierarchy in hierarchical-modular networks, and the “amount” of small-worldness in small-world networks, measured by

appropriate parameters, both seem to aid the emergence of coordination.

In general, we may note that scale-free networks and small-world networks display contrasting characteristics in terms of the evolution of coordination. The hierarchical-modular class tends to display features similar to scale-free networks, while the E-R random networks display features similar to small-world networks. However, the overarching conclusion is that topological features, qualified here by the four classes of networks, influence the evolution of coordination in social systems in non-trivial ways.

There are several implications to the results reported here. Both small-world and scale-free features are observed in real-world social systems up to various degrees. We have shown that while the emergence of coordination can be aided equally readily by both features, scale-freeness increases the sensitivity of the system to noise and time lags in information diffusion, while networks which are exclusively small-world are relatively unaffected by it. This would imply that systems, which are small-world but *not* scale-free are likely to evolve into being dominant in coordination and sustain it under difficult information diffusion conditions. This contention is further validated by the fact that the “small-worldness” itself, measured by the clustering coefficient and network diameter of the network, seems to aid the phase transition in terms of relative coordinator payoff. We noted that this has been corroborated by other studies in different game contexts. We have also experimented with several network densities (average degrees) in all classes of networks, and shown that the sparser the network is, the easier the emergence of coordination, other parameters being unchanged. Therefore, the smaller the number of games played within a network, the easier it seems for coordination to evolve as the winning strategy. These results are significant to understand the behaviour of the spatially connected social system.

Finally, it would be useful to contextualise our results against some other recent advances made in networked game theory. A number of studies have looked at the evolution of cooperation (exemplified by cooperation in Prisoners Dilemma game), and how it is influenced by graph topology. For example, reference [35] considered a range of essentially scale-free networks and showed that the heterogeneity introduced by them helps in the spread of co-operation. The relationship between payoff aspirations and cooperation was highlighted by reference [11]. Other studies have looked at the role of assortativity in the emergence of coordination (for e.g., see [12]). The study by Jiang and Perc [54] highlighted, again within the context of Prisoners Dilemma, that there exists an optimal number of inter-modular links which aids the spread of cooperation between groups. Other studies have looked at the influence of topology in hawks and doves games [53]. However, most of these studies chose one or two arbitrary classes of network topologies, and did not consider the affects of information diffusion. Our contribution in this work lies in (i) comparing all well-known classes of networks, as well as the well-mixed case, in a principled

manner, and (ii) explicitly studying the effects of information diffusion, in terms of noise as well as time-lags, in the evolution of coordination. We also chose to study coordination rather than cooperation (i.e stag hunt rather than Prisoners Dilemma), which is, while similar, perhaps less analysed in literature.

Of course, this study could be enhanced by looking at a broader set of parameters and network topologies. For example, we used only a limited range of average degrees, and mostly studied topological differences based on broad classifications rather than based on individual topological characteristics such as assortativity. The number of networks and initialisations used also could be increased. Despite these limitations, we believe the early results are indicative of some general patterns with respect to different topologies and how coordination evolves in each of them.

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