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On the Fundamentals of Angle Trisectors of a Triangle

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Abstract

Over the years of the history of elementary and advanced geometry, trisecting a given angle into three equal parts, was prominent and given more attention. Nevertheless, it is evident that there is a significant research gap of the standard angle trisectors, the lengths of the angle trisectors and the relationships amongst other standard line segments in a triangle. In this paper, we address this gap by developing a purely geometric framework, supplemented with advanced algebraic methods, to obtain closed-form expressions for internal angle trisectors in a Euclidean triangle. Using the circumcircle, similarity arguments, and Ptolemy's Theorem, we derive polynomial relations and solve the associated cubic equations explicitly through Cardano's method. The explicit determination of angle trisector lengths has not been previously available in closed form. Most approaches are trigonometric, but the trisector and related lengths were implicit or incomplete. Moreover, we present few very useful, novel, interesting lemmas, fundamental theorems, and corollaries related to two-dimensional angle trisectors in Euclidean triangles without using any trigonometric, vector algebra or complex number methods.

Keywords: Angle Trisectors, Circumcircle, Cubic Equations, Ptolemy's Theorem, Similar Triangle

Introduction

An angle trisector is a line segment or a cevian that trisects a given angle into three equal parts, drawn from the vertex of the relevant angle up to the opposite side length of the angle. The impossibility of trisecting a general Euclidean angle on a plane using only a compass and straightedge is a result from classical Greek geometry and modern algebra. It was proved impossible in the 19th century using tools from abstract algebra, particularly from field theory and Galois theory. The general proof that angles cannot be trisected with a compass and straightedge was completed by Pierre Wentzel in 1837. The angle bisectors are well known for their unique properties, such as their concurrency at the incenter and their role in defining excircles. However, deeper investigations into angle trisectors reveal equally rich and elegant geometric structures. The Morley's Trisector Theorem, (Grinberg, E.L. & Orhon, M., 2021, Dergiades) which is both striking and unexpected, has inspired a wealth of subsequent research into the geometry of trisectors and their related results. Unlike angle bisectors, trisectors do not typically enjoy simple concurrent behaviour even though one may anticipate; instead, they give rise to new points, lines, and triangles whose properties continue to fascinate mathematicians and scientists (Trisna, D., Mashadi & Gemawati, S., 2020). Nevertheless, the lengths of standard angle trisectors are undoubtedly finite and should be able to obtain explicitly. But it has been observed that obtaining the lengths of angle trisectors explicitly, was not given much attention and seemed it was intricate to find explicitly due to the limited techniques available.

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Main Results

Lemma 1
Let $ABC\Delta$be an arbitrary plane triangle and D_1 and D_2 be points on BC such that AD_1 and AD_2 are the internal angle trisectors of the angle $B\hat{A}C$. That is $B\hat{A}D_1 = D_1\hat{A}D_2 = D_2\hat{A}C = \theta$. Let $AB = c,\ BC = a$ and$AC = b,\ AD_1 = d$ and $AD_2 = e$. Then $d^2 - d.AE_1 + BD_1a - {BD_1}^2 = 0$. (Amarasinghe, I. (2012), Amarasinghe, I. (2010))

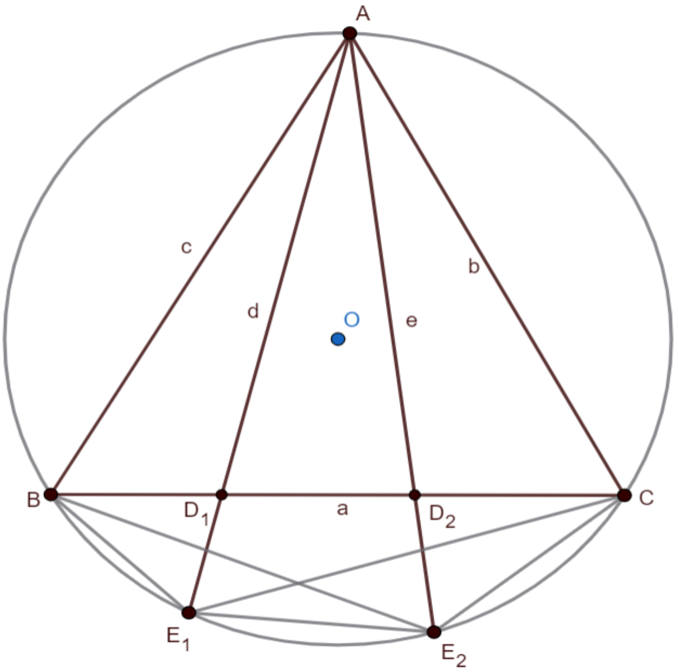

Figure 1. Angle Trisectors and Circumcircle of a Triangle

Proof of Lemma
By using the geometry of the circle, it follows that $E_1\hat{A}C = E_1\hat{B}C$ and $A\widehat{E_1}B = A\hat{C}B$. Hence $AD_1C\Delta$ and $BD_1E_1\Delta$ are similar. Then $\frac{d}{BD_1} = \frac{D_1C}{D_1E}$. Since $D_1E_1 = AE_1 - d$ and $D_1C = a - BD_1$, it follows that $d^2 - d.AE_1 + BD_1a - {BD_1}^2 = 0$.

Lemma 2
The chords composed by the extended trisectors are equal. That is $BE_1 = E_1E_2 = CE_2$. Moreover, the quadrilateral $BE_1E_2C$is a trapezoid such that $BE_2 = CE_1$, and E_1E_2 and $BC$are parallel to each other.

Proof of Lemma
Since A, B, E_1, E_2, C are on the circumference of a circle, it follows that $B\hat{A}E_1 = \theta = B\widehat{E_2}E_1 = B\hat{C}E_1 = E_2\hat{A}C = E_2\hat{B}C = E_2\widehat{E_1}C$. Thus, by the very elementary geometry it follows that $BE_1 = E_1E_2 = CE_2$ and E_1E_2 and $BC$are parallel. Since $B\widehat{E_1}E_2 = C\widehat{E_2}E_1$, it follows that $BE_1E_2\Delta$ and $E_1E_2C\Delta$ are congruent. Thus, $BE_2 = CE_1$.

Lemma 3
The length of the chord CE_2 is a function of the lengths $a, b, c, d$and is given by

$$CE_2 = \frac{ab}{2d(c^2 - b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2 + 4c^2(b + d)\right)}\right).$$

Proof of Lemma

Since $ABE_1C$is a cyclic quadrilateral, by using the Ptolemy's Theorem of cyclic quadrilaterals (Amarasinghe, I. (2024)), it follows that $a.AE_1 = b.BE_1 + c.CE_1$.

Since BE_1E_2C is also a cyclic quadrilateral, by using the Ptolemy's Theorem of cyclic quadrilaterals, it follows that

$$CE_1{}^2 = CE_2{}^2 + a.CE_2$$

since $BE_2 = CE_1$ and $BE_1 = E_1E_2 = CE_2$.

Thus, it follows that

$$AE_1 = \frac{b.CE_2 + c.\sqrt{CE_2{}^2 + a.CE_2}}{a}.$$

Observe that since E_1E_2 and BC are parallel, it follows that

$$\frac{d}{AE_1} = \frac{D_1D_2}{E_1E_2} = \frac{D_1D_2}{CE_2}.$$

Thus,

$$\frac{d}{AE_1} = \frac{a - BD_1 - D_2C}{CE_2}.$$

Observe that since ABE_1C is a cyclic quadrilateral, $BD_1E_1\Delta$ and $AD_1C\Delta$ are similar. Hence

$$\frac{BD_1}{d} = \frac{BE_1}{b} = \frac{CE_2}{b}$$

and thus

$$BD_1 = \frac{dCE_2}{b}.$$

Also, observe that $AE_1C\Delta$ and $CD_2E_2\Delta$ are similar since $B\hat{C}E_2 = E_1\hat{A}C$ and $A\widehat{E_1}C = A\widehat{E_2}C$. Thus,

$$\frac{D_2C}{b} = \frac{CE_2}{AE_1}.$$

Hence

$$D_2C = \frac{bCE_2}{AE_1}.$$

Hence it follows that

$$\frac{d}{AE_1} = \frac{a - BD_1 - D_2C}{CE_2} = \frac{a - \frac{dCE_2}{b} - \frac{bCE_2}{AE_1}}{CE_2}.$$

This leads us to

$$AE_1 = \frac{b.CE_2(b + d)}{ab - d.CE_2}.$$

Thus, it follows that

$$AE_1 = \frac{b.CE_2(b + d)}{ab - d.CE_2} = \frac{b.CE_2 + c.\sqrt{CE_2{}^2 + a.CE_2}}{a}.$$

Plugging $CE_2 = y$, a careful simplification leads us to

$$d^2(c^2 - b^2)y^2 - abd(2c^2 + bd)y + c^2a^2b^2 = 0.$$

Assuming $b \neq c$, an algebraic manipulation leads us to

$$CE_2 = \frac{ab}{2d(c^2 - b^2)}\left(2c^2 + bd \pm \sqrt{b\big(bd^2 + 4c^2(b + d)\big)}\right).$$

This completes the proof. (The case $b = c$ will be treated in due course.)

Lemma 4
The length of the extended internal angle trisector AE_1 is given by

$$AE_1 = \frac{bCE_2(b+d)}{ab-dCE_2} = -\frac{b(b+d)(\sigma K + bd + 2c^2)}{d(\sigma K + 2b^2 + bd)}$$

where $K = \sqrt{b\left(bd^2 + 4c^2(b+d)\right)}$ and $\sigma = \pm 1$.

Proof of Lemma
In the proof of the above lemma, we have proved that

$$AE_1 = \frac{b.CE_2(b+d)}{ab - d.CE_2}.$$

Hence it follows that

$$AE_1 = \frac{b.\left(\frac{ab}{2d(c^2-b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2+4c^2(b+d)\right)}\right)\right)(b+d)}{ab - d.\left(\frac{ab}{2d(c^2-b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2+4c^2(b+d)\right)}\right)\right)}.$$

Now for the sake of simplicity, put $K = \sqrt{b\left(bd^2 + 4c^2(b+d)\right)}$.
Thus,

$$AE_1 = \frac{b.\left(\frac{ab}{2d(c^2-b^2)}(2c^2 + bd \pm K)\right)(b+d)}{ab - d.\left(\frac{ab}{2d(c^2-b^2)}(2c^2 + bd \pm K)\right)}.$$

A careful simplification leads us to

$$AE_1 = -\frac{b(b+d)(-K + bd + 2c^2)}{d(-K + 2b^2 + bd)} \quad \text{or} \quad AE_1 = -\frac{b(b+d)(K + bd + 2c^2)}{d(K + 2b^2 + bd)}.$$

Now put $\sigma = \pm 1$. Hence the formulae reduces to

$$AE_1 = -\frac{b(b+d)(\sigma K + bd + 2c^2)}{d(\sigma K + 2b^2 + bd)}.$$

Lemma 5
The length of the segment CE_2 is a function of the lengths a, b, c, d and is given by

$$BD_1 = \frac{a}{2(c^2-b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2+4c^2(b+d)\right)}\right).$$

Proof of Lemma
Since in an above lemma we have proved that $\frac{BD_1}{d} = \frac{CE_2}{b}$, it follows that

$$BD_1 = \frac{dab}{2db(c^2-b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2+4c^2(b+d)\right)}\right) = \frac{dy}{b}.$$

Hence it follows that

$$BD_1 = \frac{a}{2(c^2-b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2+4c^2(b+d)\right)}\right).$$

Theorem 1 (Main Theorem)

Let $ABC\Delta$ be an arbitrary plane triangle and D_1 and D_2 be points on BC such that AD_1 andAD_2 are the internal angle trisectors of the angle $B\hat{A}C$. Then the length of the internal angle bisector $AD_1 = d$ is given by the cubic polynomial equation

$$(a^2b^2 - b^4 + 2b^2c^2 - c^4)d^3 + 3a^2bc^2d^2 + bc(a-b-c)(a-b+c)(a+b-c)(a+b+c) = 0.$$

Proof of the Main Theorem

Using the equation in Lemma 3, since $CE_2 = y$, it follows that

$$f(y) = d^2(c^2-b^2)y^2 - abd(2c^2+bd)y + a^2b^2c^2 = 0$$

and hence (provisionally assuming $b \neq c$)

$$y^2 = \frac{abd(2c^2+bd)}{d^2(c^2-b^2)}y - \frac{a^2b^2c^2}{d^2(c^2-b^2)} \quad \dots\dots\dots (1)$$

Multiplying by $y \neq 0$, it follows

$$y^3 = \frac{abd(2c^2+bd)}{d^2(c^2-b^2)}y^2 - \frac{a^2b^2c^2}{d^2(c^2-b^2)}y.$$

Replacing to y^2 from (1), it follows that

$$y^3 = \frac{abd(2c^2+bd)}{d^2(c^2-b^2)}\left(\frac{abd(2c^2+bd)}{d^2(c^2-b^2)}y - \frac{a^2b^2c^2}{d^2(c^2-b^2)}\right) - \frac{a^2b^2c^2}{d^2(c^2-b^2)}y.$$

That leads us to

$$y^3 = \frac{a^2b^2}{d^2(c^2-b^2)^2}\left((2c^2+bd)^2 - c^2(c^2-b^2)\right)y - \frac{a^3b^3c^2(2c^2+bd)}{d^3(c^2-b^2)^2}.$$

Then from Lemma 1, $d^2 - d.AE_1 + BD_1a - {BD_1}^2 = 0$. Thus

$$d^2 - d.\left(\frac{by(b+d)}{ab-dy}\right) + \left(\frac{dy}{b}\right)a - \left(\frac{dy}{b}\right)^2 = 0.$$

This leads us to $g(y) = d^2y^3 - 2abdy^2 + y(a^2b^2 - b^3d - b^2d^2 - b^4) + ab^3d = 0.$

Now by substituting $y^2$and y^3obtained earlier, for $g(y)$, and after simplifying it follows that

$$g(y) = \left(\frac{a^2b^2}{(c^2-b^2)^2}\left((2c^2+bd)^2 - c^2(c^2-b^2)\right)y - \frac{a^3b^3c^2(2c^2+bd)}{d(c^2-b^2)^2}\right)$$
$$-2ab\left(\frac{ab(2c^2+bd)}{(c^2-b^2)}y - \frac{a^2b^2c^2}{d(c^2-b^2)}\right) + y(a^2b^2 - b^3d - b^2d^2 - b^4) + ab^3d = 0.$$

Hence it follows that

$$g(y) = \left(\frac{a^2b^2}{(c^2-b^2)^2}\left((2c^2+bd)^2 - c^2(c^2-b^2)\right) - \frac{2a^2b^2(2c^2+bd)}{(c^2-b^2)} + (a^2b^2 - b^3d - b^2d^2 - b^4)\right)y$$
$$+\left(\frac{2a^3b^3c^2}{d(c^2-b^2)} - \frac{a^3b^3c^2(2c^2+bd)}{d(c^2-b^2)^2} + ab^3d\right) = 0.$$

Now we eliminate $\boldsymbol{y}$by using the above linear equation and the quadratic equation $\boldsymbol{f(y)}$ (by keeping the coefficient (c^2-b^2) in the polynomials so that $b = c$ case is generalized) and after a careful factorization, we find the resultant $R_{x,y}$ of the polynomials $f(y)$and $g(y)$, and since they both have a common root, it follows that

$$R_{x,y} = a^2b^7d^6(b+d)^2\big((a^2b^2 - b^4 + 2b^2c^2 - c^4)d^3 + 3a^2bc^2d^2 + (a^4bc^2 - 2a^2b^3c^2 - 2a^2bc^4 + b^5c^2 - 2b^3c^4 + bc^6)\big) = 0.$$

Since clearly $a, b, d, b + d > 0$, it follows that the required polynomial is given by

$$(a^2b^2 - b^4 + 2b^2c^2 - c^4)d^3 + 3a^2bc^2d^2 + (a^4bc^2 - 2a^2b^3c^2 - 2a^2bc^4 + b^5c^2 - 2b^3c^4 + bc^6) = 0.$$

That is,

$$(a^2b^2 - b^4 + 2b^2c^2 - c^4)d^3 + 3a^2bc^2d^2 + bc(a-b-c)(a-b+c)(a+b-c)(a+b+c) = 0.$$

Now in order to obtain the zeros of the above polynomial equation, we use the standard Cardona's Method without proof. Put

$A = a^2b^2 - b^4 + 2b^2c^2 - c^4$,

$B = 3a^2bc^2$ and

$C = bc(a-b-c)(a-b+c)(a+b-c)(a+b+c) = a^4bc^2 - 2a^2b^3c^2 - 2a^2bc^4 + b^5c^2 - 2b^3c^4 + bc^6$

where $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$are constants.

Remark 1. Since in an arbitrary triangle, $a + b + c > 0$, $a + b > c$, $a + c > b$and $a < b + c$, it follows that $C < 0$. Also, clearly $B > 0$.

Case 1. Suppose $\boldsymbol{A \neq 0}$.

Now put $\alpha = \frac{B}{A}$ and $\lambda = \frac{C}{A}$. Also let $d = y - \frac{\alpha}{3}$ and $y = U + V$. Then the above cubic is depressed to

$$y^3 - \frac{\alpha^2 y}{3} + \frac{2\alpha^3}{27} + \lambda = 0.$$

Now put $p = \frac{-\alpha^2}{3}$and $q = \frac{2\alpha^3}{27} + \lambda$. Then the discriminant Δ_y of the cubic is given by $\Delta_y = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$. Replacing for $p$and q in terms of A, B and C, and simplifying further, we yield that

$$\Delta_y = \frac{C(27A^2C + 4B^3)}{108A^4}.$$

Then according to the standard Cardona's Method,

$$U = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta_y}} \text{ and } V = \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta_y}}\ .$$

Thus, the roots of the cubic polynomial equation of d, are given by

$$d_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta_y}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta_y}} - \frac{B}{3A} \text{ or}$$

$$d_2 = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta_y}}\,\omega + \omega^2 \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta_y}} - \frac{B}{3A} \quad \text{or}$$

$$d_3 = \omega^2 \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta_y}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta_y}}\,\omega - \frac{B}{3A}$$

where $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

Observe that if $\Delta_y > 0$, then the cubic has one real root (that is d_1) and the rest of the roots are complex conjugate pairs.

If $\Delta_y = 0$ then the cubic has a double root which is real.

If $\Delta_y < 0$, then the cubic has three distinct real roots which can be given by a standard trigonometric formula according to the Cardona's method. Then $y_k = 2\sqrt{-\frac{p}{3}} cos\left(\frac{1}{3} cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right)$ for $k = 0,1,2$. Thus $d_k = y_k - \frac{B}{3A} = 2\sqrt{-\frac{p}{3}} cos\left(\frac{1}{3} cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right) - \frac{B}{3A}$ for $k = 0,1,2$.

Observe that we only choose the positive root for the value of d.

Case 2. Suppose $A = 0$.

Then, it follows that $d = \pm\sqrt{-\frac{C}{B}}$ assuming $B \neq 0$. Moreover, even if $B = 0$, then you have $C = 0$ for a solution. Moreover, $(b^2 - c^2)^2 = a^2 b^2$.

Theorem 2

Let $ABC\Delta$ be an arbitrary plane triangle and D_1 and D_2 be points on BCsuch that AD_1 and AD_2 are the internal angle trisectors of the angle $B\hat{A}C$. Then the value of the internal angle trisector $AD_2 = e$ is given by $e = -\frac{cd(\sigma K + 2b^2 + bd)}{(b+d)(\sigma K + bd + 2c^2)}$ where $K = \sqrt{b\left(bd^2 + 4c^2(b+d)\right)}$ and $\sigma = \pm 1$.

Proof of the Theorem

Observe that since $A\hat{C}B = A\widehat{E_1}B$ and $B\hat{A}D_1 = C\hat{A}D_2 = \theta$, it follows that $ABE_1\Delta$ and $ACD_2\Delta$ are similar. Thus, it follows $\frac{e}{c} = \frac{b}{AE_1}$. Hence $e = \frac{cb}{AE_1}$. Hence by using the above obtained length of AE_1, it follows that $e = -\frac{cd(\sigma K + 2b^2 + bd)}{(b+d)(\sigma K + bd + 2c^2)}$ where $K = \sqrt{b\left(bd^2 + 4c^2(b+d)\right)}$ and $\sigma = \pm 1$.

Remark 2 (for a future development). Observe that the segment D_2C is obtained by $\frac{D_2C}{e} = \frac{CE_2}{c}$ since $AD_2C\Delta$ and $ABE_1\Delta$are similar. Thus

$$D_2C = \frac{eCE_2}{c} = \left(\frac{-\frac{cd(\sigma K + 2b^2 + bd)}{(b+d)(\sigma K + bd + 2c^2)}}{c}\right)\frac{ab}{2d(c^2 - b^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2 + 4c^2(b+d)\right)}\right)$$

$$D_2C = \frac{-ab(\sigma K + 2b^2 + bd)}{2(c^2 - b^2)(b+d)(\sigma K + bd + 2c^2)}\left(2c^2 + bd \pm \sqrt{b\left(bd^2 + 4c^2(b+d)\right)}\right).$$

Theorem 3

If $ABC\Delta$is isosceles such that $b = c$, then it follows that $d = e$ and the possible value for the length $d$is given by $d = b\left(2\cos\left(\frac{\phi}{3}\right) - 1\right)$ where $\phi = \cos^{-1}\left(\frac{2b^2 - a^2}{2b^2}\right)$.

Proof of Theorem

Assume that $ABC\Delta$ is isosceles such that $b = c$. Hence by the elementary geometry, it follows that $A\hat{B}C = A\hat{C}B$. Since $B\hat{A}D_1 = D_2\hat{A}C = \theta$, we yield $AD_1B\Delta$ and $AD_2C\Delta$ are congruent. Thus, $d = e$. Then by the cubic polynomial of $d$in the main theorem, after some simplification, the cubic is reduced to $(d + b)(d^3 + 3bd^2 + a^2b - 4b^3) = 0$. Since $b + c > 0$, it follows $d^3 + 3bd^2 + b(a^2 - 4b^2) = 0$. Then by applying the general formula we obtained for $d$earlier using Cardona's method, and simplifying it further with $b = c$, it is not difficult to obtain the above mentioned lengths for d.

Observe that according to the inequality $a^2 - 4b^2 \geq 0$ or $a^2 - 4b^2 < 0$, we can categorize the roots whether they are positive, negative or complex as mentioned in the above theorem. Now assume that $a^2 - 4b^2 = 0$. Then $a = 2b$. Since $b = c$, it follows that $b + c = a$. But from the triangle inequality, $b + c > a$. This is a contradiction. Hence it follows that $a^2 \neq 4b^2$. Also, assume that $a > 2b$. That is $a > b + c$ since $b = c$. This is a contradiction, since $a < b + c$by the triangle inequality.

Hence the possible real roots are obtained, only if $a^2 - 4b^2 < 0$. That is, if $a < 2b$. That is then $\Delta_d < 0$. Then by the main Theorem, the possible roots are obtained by

$$d_k == 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - \frac{2\pi k}{3}\right) - \frac{B}{3A} \quad \text{for } k = 0,1,2.$$

Thus $p = \frac{-\alpha^3}{3} = -3b^2$ and $q = \frac{2\alpha^3}{27} + \lambda = \frac{a^2 - 2b^2}{2}$ since in the standard depressed cubic in the proof of main Theorem, they are the coefficients of $y$and the constant term. Hence it follows that after simplifying the trigonometric solution carefully,

$$d_k = b\left(2\cos\left(\frac{\cos^{-1}\left(\frac{2b^2 - a^2}{2b^2}\right)}{3} - \frac{2\pi k}{3}\right) - 1\right) \text{ for } k = 0,1,2$$

Now observe that since $0 < a < 2b$, if $k = 1$and $k = 2$, then it is easy to see that the roots $d_1$and $d_2$are both non-positive. Since $d > 0$, the only solution (positive) is given at $k = 0$. Thus

$$d_0 = b\left(2\cos\left(\frac{cos^{-1}\left(\frac{2b^2 - a^2}{2b^2}\right)}{3}\right) - 1\right).$$

Now put $\phi = cos^{-1}\left(\frac{2b^2-a^2}{2b^2}\right)$. Then the only positive solution is given by $d = d_0 = b\left(2\cos\left(\frac{\phi}{3}\right) - 1\right)$. This completes the proof.

Theorem 4

If $ABC\Delta$is equilateral, then $d = e$ and the possible value for the length $d$is given by $d = a\left(2\cos\left(\frac{\pi}{9}\right) - 1\right)$.

Proof of Theorem

Proof is trivial from Theorem 3 by putting $b = a$, and since $\phi = cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ only, as for all the other possible values for ϕ, $d$is negative. Using sin formula also, this result is easily validated.

Remark 3. Note that it is easily followed by the congruence of triangles that $b = c$ **if and only if** $d = e$.

Theorem 5

The segments BD_1, BD_2, D_1C, D_2C generated on BC are related as $\left(\frac{BD_1}{D_1C}\right)\left(\frac{BD_2}{D_2C}\right) = \frac{c^2}{b^2}$.

Proof of the Theorem

Due to the geometry of the circle (angles of the same chords), $ABD_1\Delta$ and $AE_2C\Delta$ are similar. Hence $\frac{BD_1}{E_2C} = \frac{c}{AE_2}$. Also $AD_1C\Delta$ and $ABE_2\Delta$ are similar. Hence $\frac{D_1C}{BE_2} = \frac{b}{AE_2}$. Thus $\frac{BD_1}{D_1C} = \left(\frac{E_2C}{BE_2}\right)\left(\frac{c}{b}\right)$.

Also $BD_2E_2\Delta$ and $ABE_1\Delta$ are similar. Hence $\frac{BD_2}{c} = \frac{BE_2}{AE_1}$. And $AD_2C\Delta$ and $ABE_1\Delta$ are similar. Hence $\frac{D_2C}{BE_1} = \frac{b}{AE_1}$. Thus $\frac{BD_2}{D_2C} = \left(\frac{c}{b}\right)\left(\frac{BE_2}{E_2C}\right)$. Hence by multiplying the given segment ratios, it follows that $\left(\frac{BD_1}{D_1C}\right)\left(\frac{BD_2}{D_2C}\right) = \frac{c^2}{b^2}$.

Corollary 1

There is no Euclidean plane triangle such that two internal angle trisectors and their adjacent side lengths are all equal to each other.

Proof of Corollary

On the contrary assume that there is a Euclidean plane triangle such that two internal angle trisectors and their adjacent side lengths are all equal to each other. Then it follows that $b = c = d = e$. Using the proof of Theorem 3, it follows that $d^3 + 3bd^2 + b(a^2 - 4b^2) = 0$. Then $b^3 + 3bb^2 + b(a^2 - 4b^2) = 0$. Hence $a = 0$. This is a contradiction since $a > 0$. Thus, our assumption is false. That is, there is no Euclidean triangle such that two internal angle trisectors and their adjacent side lengths are all equal to each other. This completes the proof.

Conclusions

This paper tries to provide a systematic treatment of angle trisectors in a Euclidean triangle to some considerable extent, addressing a longstanding gap in the explicit determination of their lengths and related identities. By employing a purely geometric approach supplemented with algebraic methods, we established novel lemmas, theorems, and corollaries that highlight the rich structure generated by trisectors. In particular, we derived closed-form relations for trisector lengths, demonstrated their interplay with classical results with similar triangles and Ptolemy's theorem, and clarified their special behaviour in isosceles and equilateral cases. The results illustrate how advanced algebra, notably the solution of cubic equations via Cardano's method, can successfully complement pure geometry in resolving intricate problems. These findings deepen the understanding of angle trisectors and open avenues for further study in cevian geometry and advanced Euclidean geometry.

References

Amarasinghe, I. (2024). An alternative proof of Ptolemy's theorem and its variations. *SLIIT Journal of Humanities and Sciences*, *4*(1), 1–11.

Amarasinghe, I. (2012). On the standard lengths of angle bisectors and the angle bisector theorem. *Global Journal of Advanced Research on Classical and Modern Geometries*, 15–27.

Amarasinghe, I. (2010). Advanced plane geometry research 1. *Proceedings of the 66th Annual Sessions of the Sri Lanka Association for the Advancement of Science*, 66, 77.

Dergiades, N., & Tran Quang Hung. (2020). On some extensions of Morley's trisector theorem. *arXiv*. https://arxiv.org/abs/2005.08723.

Grinberg, E. L., & Orhon, M. (2021). Morley trisectors and the law of sines with reflections. *The American Mathematical Monthly, 128*(2), 163–167. https://doi.org/10.1080/00029890.2021.1845559.

Trisna, D., Mashadi, & Gemawati, S. (2020). Angle trisector in the triangle. *IOSR Journal of Mathematics, 16*(5), 11–18.